2025年秋，有限元方法II，上机作业

截至时间：2026/01/07，晚上12点

要求:

- 用TeX写上机报告(中英文均可)，包含必要的数值结果讨论，页数上限20.
- 本次上机作业不限制程序语言与软件包.
- 截止时间前将程序和上机报告的源码发送至snwu@math.pku.edu.cn

Consider the following mixed formulation of the Poisson equation

$$\begin{cases} \boldsymbol{p} - \nabla u = 0 & \text{in } \Omega \subset \mathbb{R}^2, \\ -\mathrm{div}\, \boldsymbol{p} = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Use the mixed finite element spaces $\mathrm{RT}_k - \mathcal{P}_k^{-1}$ and $\mathrm{BDM}_{k+1} - \mathcal{P}_k^{-1}$ to solve (**??**). Here, the source term f and boundary data g are derived from the exact solution u. Perform the computations on *quasi-uniform* meshes.

Problem 1 (convergence rates). Choose $\Omega = (-1,1)^2$ and a *smooth* solution u. Report the errors for $\boldsymbol{p}$ in the $\boldsymbol{H}(\mathrm{div})$ norm and L^2 norm, as well as the errors for u in the L^2 norm.

Problem 2 (post-processing). For the aforementioned setting, implement the post-processing of u and report the errors in the L^2 norm.

Problem 3 (singular solution & pollution effect). Consider a non-convex domain defined as

$$\Omega := \{(x,y) \in (-1,1)^2 : 0 < \theta < \pi/\beta\}, \quad \text{where } \frac{1}{2} \le \beta < 1.$$

Set the exact solution to be

$$u = r^\beta \sin(\beta\theta). \tag{2}$$

- Report the errors for $\boldsymbol{p}$ in the $\boldsymbol{H}(\mathrm{div})$ norm and L^2 norm, as well as the errors for u in the L^2 norm for different values of β.

- For the subdomain Ω' obtained by removing the region near the reentrant corner, for example, $\Omega' = \{(x, y) \in \Omega \setminus [-0.5, 0.5]^2\}$. Report the $L^2(\Omega')$ errors of u.

Remark: The case in which $k = 0$ (RT_0-$\mathcal{P}_0^{-1}$ and BDM_1-$\mathcal{P}_0^{-1}$) is required. At least one high-order case (e.g., RT_1-$\mathcal{P}_1^{-1}$ or BDM_2-$\mathcal{P}_1^{-1}$) is required.

Problem 4 (equivalence with the Crouzeix–Raviart element). On the same mesh $\mathcal{T}_h$, implement the CR finite element method and the RT_0-$\mathcal{P}_0^{-1}$ mixed method to solve the Poisson equation. Denote the numerical solutions by u_h^{CR} and $(\boldsymbol{\sigma}_h, u_h)$, respectively.

Let f be a piecewise constant function. Numerically verify that

$$\boldsymbol{\sigma}_h|_T = \nabla u_h^{\mathrm{CR}}|_T - \frac{f}{2}(\boldsymbol{x} - \boldsymbol{x}_T), \quad \forall T \in \mathcal{T}_h.$$

where $\boldsymbol{x}_T$ denotes the barycenter of each element.