

2025秋，有限元方法II，作业5

交作业时间：2025/12/18

1. Given any $\epsilon > 0$, let $X = H(\text{curl}) \cap \underline{H}^{1/2+\epsilon}$. Given a Lipschitz domain Ω , show that there exists $\delta(\epsilon, \Omega) > 0$ so that $\text{curl}X(\Omega) \subset \underline{L}^{2+\delta(\epsilon, \Omega)}(\Omega)$. Let $W = H(\text{div}) \cap \underline{L}^{2+\delta(\epsilon)}$. Choose one of the following to prove the commutative diagram:

$$\begin{array}{ccc} X(K) & \xrightarrow{\text{curl}} & W(K) \\ \Pi_k^N \downarrow & & \downarrow \Pi_k^{RT} \\ N_k(K) & \xrightarrow{\text{curl}} & RT_k(K) \end{array} \quad \begin{array}{ccc} X(K) & \xrightarrow{\text{curl}} & W(K) \\ \Pi_{k+1}^{NC} \downarrow & & \downarrow \Pi_k^{RT} \\ NC_{k+1}(K) & \xrightarrow{\text{curl}} & RT_k(K) \end{array}$$

Find the similar version for $BDM_k(K)$ (no need to show the proof).

2. Show that the following two inequalities are equivalent

$$\|p\|_{L^2(\Omega)} \lesssim \|p\|_{H^{-1}(\Omega)} + \sum_{i=1}^d \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(\Omega)} \quad \forall p \in L^2(\Omega), \quad (1)$$

$$\|p\|_{L^2(\Omega)} \lesssim \sum_{i=1}^d \left\| \frac{\partial p}{\partial x_i} \right\|_{H^{-1}(\Omega)} \quad \forall p \in L_0^2(\Omega). \quad (2)$$

3. Let Ω be a connected domain with a Lipschitz boundary. Assume that $\Gamma_D \subset \partial\Omega$ satisfies $\text{meas}(\Gamma_D) \neq 0$. Show that

$$\|v\|_{\underline{H}^1(\Omega)} \lesssim \|\epsilon(v)\|, \quad \forall v \in \underline{H}_D^1(\Omega),$$

where $\underline{H}_D^1(\Omega) := \{v \in \underline{H}^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$.

4. For the Stokes pair $\mathcal{P}_1^{\text{CR}}\text{-}\mathcal{P}_0^{-1}$, prove that it satisfies the following discrete inf-sup condition:

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{(\text{div}_h v_h, q_h)}{\|v_h\|_{1,h} \|q_h\|_{L^2}} \gtrsim 1,$$

where div_h denotes the piecewise divergence, and $\|\cdot\|_{1,h}$ represents the piecewise H^1 -norm. For the aforementioned element, provide the error estimate for the following numerical scheme ($\nu = 1$):

$$\begin{cases} 2(\epsilon_h(u_h), \epsilon_h(v_h)) - (\text{div}_h v_h, p_h) = (\underline{f}, v_h) & \forall v_h \in V_h, \\ -(\text{div}_h u_h, q_h) = 0 & \forall q_h \in Q_h. \end{cases}$$

5. Let $V = \underline{H}_0^1(\Omega)$. Consider $V^0 = \{\underline{v} \in V \mid \int_T \operatorname{div} \underline{v} \, dx = 0, \forall T \in \mathcal{T}_h\}$, define $\Pi_2 : V^0 \rightarrow B(\operatorname{grad} Q_h)$ by

$$\begin{cases} \Pi_2 \underline{v}|_T \in B(\operatorname{grad} Q_h)|_T, \\ \int_T \operatorname{div}(\Pi_2 \underline{v} - \underline{v}) \, dx = 0, \quad \forall q_h \in Q_h|_T, \end{cases} \quad \forall T \in \mathcal{T}_h.$$

Show that $\|\Pi_2 \underline{v}\|_1 \lesssim \|\underline{v}\|_1$ for $\underline{v} \in V^0$.

6. Consider the Stokes problem with homogeneous Dirichlet boundary condition:

$$\begin{aligned} -\Delta \underline{u} + \nabla p &= \underline{f} \quad \text{in } \Omega, \\ -\operatorname{div} \underline{u} &= 0 \quad \text{in } \Omega, \\ \underline{u}|_{\partial\Omega} &= \underline{0}. \end{aligned}$$

Let $V = \underline{H}_0^1(\Omega)$ and $Q = L_0^2(\Omega)$. Given a stable Stokes pair $V_h \times Q_h \subset V \times Q$, we can obtain the following energy estimate

$$\|\underline{u} - \underline{u}_h\|_{H^1} + \|p - p_h\|_{L^2} \lesssim \inf_{\underline{v}_h \in V_h} \|\underline{u} - \underline{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}.$$

Assume further the approximation property of $V_h \times Q_h$:

$$\begin{aligned} \inf_{\underline{v}_h \in V_h} \|\underline{z} - \underline{v}_h\|_{H^1} &\lesssim h \|\underline{z}\|_{H^2} \quad \forall \underline{z} \in \underline{H}^2(\Omega), \\ \inf_{q_h \in Q_h} \|r - q_h\|_{L^2} &\lesssim h \|r\|_{H^1}, \quad \forall r \in H^1(\Omega). \end{aligned}$$

Duality argument: Find appropriate regularity assumption of the dual problem:

$$\begin{aligned} -\Delta \underline{z} + \nabla r &= \underline{\theta} \quad \text{in } \Omega, \\ \operatorname{div} \underline{z} &= 0 \quad \text{in } \Omega, \\ \underline{z}|_{\partial\Omega} &= \underline{0}. \end{aligned}$$

so that one can obtain the L^2 estimate of $\underline{u}$:

$$\|\underline{u} - \underline{u}_h\|_{L^2} \lesssim h \left(\inf_{\underline{v}_h \in V_h} \|\underline{u} - \underline{v}_h\|_{H^1} + \inf_{q_h \in Q_h} \|p - q_h\|_{L^2} \right).$$

7. Assume that the pair $V \times Q$ and its conforming discrete subspace $V_h \times Q_h$ both satisfy the inf-sup condition, namely,

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} = \beta > 0, \quad \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} = \beta_h \geq \beta_0 > 0.$$

Prove that there exists a Fortin operator $\Pi_h : V \rightarrow V_h$ such that

$$b(v, q_h) = b(\Pi_h v, q_h) \quad \forall q_h \in Q_h, \quad \text{and} \quad \|\Pi_h\| \leq C.$$