2025秋，有限元方法II，作业4

交作业时间: 2025/11/27

1. On a triangle K, define the space

$$\mathbb{H}_k(K) := \{\underset{\sim}{q} \in \underset{\sim}{\mathcal{P}}_k(K) \mid \mathrm{div}\underset{\sim}{q} = 0 \quad \text{and} \quad \underset{\sim}{q} \cdot \underset{\sim}{n}|_{\partial K} = 0\}.$$

   Show that in 2D, $\mathbb{H}_k(K) = \mathrm{curl}(b_K \mathcal{P}_{k-2}(K))$. Here, b_K is the bubble function on K, and the curl operator acts on a scalar function f as $\mathrm{curl}\, f = (-\partial_y f, \partial_x f)^T$. (Hint: You can directly utilize the dimension formula for $\mathbb{H}_k(K)$ discussed in class.)

2. Prove the following identity:

$$\underset{\sim}{x}(\nabla \cdot \underset{\sim}{w}) - \nabla \times (\underset{\sim}{x} \times \underset{\sim}{w}) = (k+2)\underset{\sim}{w}, \quad \forall \underset{\sim}{w} \in \underset{\sim}{\mathcal{H}}_k,$$

   where $\underset{\sim}{\mathcal{H}}_k$ denotes the space of vector-valued homogeneous polynomials of degree k.

3. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. For any $\underset{\sim}{v} \in (L^2(\Omega))^2$, show that there exists $\psi \in H_0^1(\Omega)$ and $\phi \in H^1(\Omega)$, such that

$$\underset{\sim}{v} = \nabla\psi + \mathrm{curl}\phi,$$

   and

$$\|\nabla\phi\|_{L^2} + \|\nabla\psi\|_{L^2} \lesssim \|\underset{\sim}{v}\|_{L^2}.$$

   (Hint: $R(\mathrm{curl}) = N(\mathrm{div})$ on simply connected domain.)

4. Let $\Omega \subset \mathbb{R}^3$ be the half-space defined by $x_3 < 0$ and Γ be the space $x_3 = 0$. Given a vector field $\underset{\sim}{\chi} = (\chi_1, \chi_2, \chi_3)^T \in H(\mathrm{curl}, \Omega)$, show that

$$\mathrm{div}_\Gamma \mathrm{Tr}\underset{\sim}{\chi} = \mathrm{curl}\underset{\sim}{\chi} \cdot \underset{\sim}{n}|_\Gamma,$$

   where $\mathrm{Tr}\chi = \underset{\sim}{\chi} \times \underset{\sim}{n}$.

5. Two results about vector-valued polynomials in three-dimensional space:

   (a) Let $\underset{\sim}{w} \in \underset{\sim}{\mathcal{H}}_k$. Define

$$\underset{\sim}{\eta} = \frac{\nabla \times \underset{\sim}{w}}{k+1}, \quad \mu = \frac{\underset{\sim}{x} \cdot \underset{\sim}{w}}{k+1}.$$

   Prove the identity: $-\underset{\sim}{x} \times \underset{\sim}{\eta} + \nabla\mu = \underset{\sim}{w}$.

(b) Show that for any vector polynomial $\underset{\sim}{p}_k \in \underset{\sim}{\mathcal{P}}_k$, there exists a decomposition

$$\underset{\sim}{p}_k = \underset{\sim}{w}_k + \nabla\theta,$$

where $\underset{\sim}{w}_k \in \underset{\sim}{\mathcal{P}}_{k-1} + \underset{\sim}{x} \times \underset{\sim}{\mathcal{P}}_{k-1}$ and $\theta \in \mathcal{P}_{k+1}$.

6. Show that for any vector polynomial $\underset{\sim}{p}_k \in \underset{\sim}{\mathcal{P}}_k$, there exists a decomposition

$$\underset{\sim}{p}_k = \underset{\sim}{w}_k + \nabla \times \underset{\sim}{\theta},$$

where $\underset{\sim}{w}_k \in \underset{\sim}{\mathcal{P}}_{k-1} + \underset{\sim}{x}\mathcal{P}_{k-1}$ and $\underset{\sim}{\theta} \in \underset{\sim}{\mathcal{P}}_{k+1}$.

7. Using the above results, prove the unisolvence of the second kind Nédélec element: $\mathrm{NC}_k(K) = \underset{\sim}{\mathcal{P}}_k(K)$, with degrees of freedom

$$\begin{aligned} \int_e \underset{\sim}{v} \cdot \underset{\sim}{t} p_k \mathrm{d}\ell, \qquad & \forall p_k \in \mathcal{P}_k(e), \\ \int_F (\underset{\sim}{v} \times \underset{\sim}{n}) \cdot \underset{\sim}{\phi}_{k-1} \mathrm{d}s, \qquad & \forall \underset{\sim}{\phi}_{k-1} \in \mathrm{RT}_{k-2}(F), \\ \int_K \underset{\sim}{v} \cdot \underset{\sim}{q}_{k-2} \mathrm{d}x, \qquad & \forall \underset{\sim}{q}_{k-2} \in \mathrm{RT}_{k-3}(K). \end{aligned}$$

8. Let $v = \mathcal{F}(\hat{v}) := \hat{v}(F^{-1}(x))$, and $\underset{\sim}{q} = \mathcal{G}(\underset{\sim}{\hat{q}})$, where $F(\hat{x}) = B\hat{x} + b_0$ is an affine mapping. Show that

$$\begin{aligned} \int_K \underset{\sim}{q} \cdot \mathrm{grad} v \, \mathrm{d}x &= \int_{\hat{K}} \underset{\sim}{\hat{q}} \cdot \widehat{\mathrm{grad}} \hat{v} \, \mathrm{d}\hat{x}, \\ \int_K v \, \mathrm{div} \underset{\sim}{q} \, \mathrm{d}x &= \int_{\hat{K}} \hat{v} \, \widehat{\mathrm{div}} \underset{\sim}{\hat{q}} \, \mathrm{d}\hat{x}, \\ \int_{\partial K} \underset{\sim}{q} \cdot \underset{\sim}{n} v \, \mathrm{d}s &= \int_{\partial \hat{K}} \underset{\sim}{\hat{q}} \cdot \underset{\sim}{\hat{n}} \hat{v} \, \mathrm{d}\hat{s}. \end{aligned}$$