2025秋，有限元方法II，作业3

交作业时间：2025/11/13

The Mathematical Theory of Finite Element Methods:

- Chapter 9: 9.x.19
- Chapter 10: 10.x.7, 10.x.11

Supplementary Questions:

1. Conduct the duality argument for the IPDG scheme

$$a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

Here, $V_h = \{v \in L^2(\Omega) : v|_T \in \mathcal{P}_1(T)\ \forall T \in \mathcal{T}_h\}$, and

$$\begin{aligned} a_h(u_h, v_h) := &\sum_{T\in\mathcal{T}_h} \int_T \nabla u_h \cdot \nabla v_h \mathrm{d}x - \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla u_h\}\!\} \cdot [\![v_h]\!] \mathrm{d}s \\ &- \sum_{e\in\mathcal{E}_h} \int_e \{\!\{\nabla v_h\}\!\} \cdot [\![u_h]\!] \mathrm{d}s + \sum_{e\in\mathcal{E}_h} \eta h_e^{-1} \int_e [\![u_h]\!] \cdot [\![v_h]\!] \mathrm{d}s. \end{aligned}$$

2. In adaptive a posteriori error estimation, define the element residual as $R_A|_T := f + \nabla\cdot(\alpha\nabla u_h)|_T$, and the jump residual as $R_J|_e = -[\alpha\nabla u_h]|_e$. Let $\overline{R_J}$ denote the average value of R_J on the edge. The patch of elements sharing the edge e is denoted by ω_e. Prove that the following local bound holds:

$$\|\overline{R_J}\|_{L^2(e)} \lesssim h_e^{1/2}\|R_A\|_{L^2(\omega_e)} + h_e^{-1/2}\|R\|_{H^{-1}(\omega_e)} + \|R_J - \overline{R_J}\|_{L^2(e)}.$$

3. Let V, Q be Banach spaces, $B : V \to Q'$ be a bound linear operator, $Z := N(B)$. For any subspace $S \subset V$, define

$$S^\circ := \{f \in V' \mid \langle f, v\rangle = 0,\ \forall v \in S\}.$$

For any subspace $F \subset V'$, define

$${}^\circ F := \{v \in V \mid \langle f, v\rangle = 0,\ \forall f \in F\}.$$

   - Show that S° and ${}^\circ F$ are closed.

- Show that ${}^{\circ}(S^{\circ}) = S$ if and only if S is closed in V; And $({}^{\circ}F)^{\circ} = F$ if and only if F is closed in V'.
- Show that ${}^{\circ}R(B') = Z$.
- Show that $R(B') = Z^{\circ}$ if and only if $R(B')$ is closed in V'.

4. Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \to H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then, the following indentity holds

$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}.$$