

EXPLICIT EXPOSURE OF HAAR MEASURE

Yixuan Huang
Peking University

Oliver Jenkinson*
Queen Mary University of London

Zhiqiang Li
Peking University

Hang Zhao
Neijiang Normal University

ABSTRACT. We solve the problem, posed some twenty years ago, of explicitly constructing a continuous function whose unique maximizing measure for the doubling map is Lebesgue measure. This is a special case of a more general result: given a nontrivial compact metrizable abelian group and a continuous endomorphism for which normalised Haar measure is ergodic, we explicitly construct a continuous function on the group for which Haar measure is the unique invariant maximizing measure. The function is the uniform limit of a recursively defined sequence of trigonometric polynomials with rational coefficients; every parameter of the recursion is given by an explicit formula, every step is exact, and the rate of convergence is explicit. For expanding circle endomorphisms, ergodic toral endomorphisms, and the one-sided full shift, the construction can moreover be arranged so that the function is given by a Fourier series that converges uniformly in the classical frequency order, each of whose coefficients is rational and computable exactly by a finite procedure.

We also show that no continuous function whose unique maximizing measure is Haar measure has absolutely summable Fourier coefficients, so that no such function lies in the Wiener algebra. For expanding maps of the circle, the Fourier coefficients of a continuous function with a unique maximizing measure of full support admit no closed form, in the sense that they satisfy no linear recurrence with polynomial coefficients; this follows from a result of independent interest, that a continuous function on the circle whose Fourier coefficients satisfy such a recurrence is Hölder continuous.

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1. INTRODUCTION

1.1. Ergodic optimization and the inverse problem. For any measure m that is ergodic for a topological dynamical system $T: X \rightarrow X$ (a continuous self-map of a compact metric space (X, d)), it is known (see [31]) that there exist continuous functions $f: X \rightarrow \mathbb{R}$ such that $\int f dm$ is *strictly* larger than $\int f d\mu$ for all other T -invariant probability measures μ . In other words, every ergodic measure is the unique f -maximizing measure for some continuous f . However, the argument of [31] does not exhibit such an f *explicitly*.

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*Corresponding author: o.jenkinson@qmul.ac.uk.

If m is *strictly ergodic*, i.e., if $\text{supp } m$ carries no invariant measure other than m , then explicit examples of such continuous f can be readily found, by arranging that f attains its maximum value precisely on $\text{supp } m$, for example by setting $f(x) := -d(x, \text{supp } m)$. If m is not strictly ergodic, then this construction is unavailable, and the problem of exhibiting an explicit continuous f whose unique maximizing measure is m is considerably more delicate. To make the question concrete, it was posed as Problem 3.9 in [30] (see also the discussion around [31, Corollary 1]) in the particular case of the doubling map $T(x) = 2x \pmod{1}$ on the circle $X = \mathbb{R}/\mathbb{Z}$, with m Lebesgue measure; in this form, the problem has been publicised in lectures and problem sessions over the past two decades¹.

Ergodic optimization has developed considerably since [30]; for surveys see [3, 32]. Problem 3.9 was one of several problems posed in [30], and it may be useful to record the status of the others. The question of *zero temperature limits*, whether the equilibrium states of tf converge as $t \rightarrow +\infty$ [30, footnote 16], was answered negatively by Chazottes and Hochman [10]. The question of *typical periodic optimization*, whether a generic Lipschitz function has a periodic maximizing measure [30, Conjecture 4.11], was answered affirmatively for expanding maps by Contreras [11], and subsequently in wider settings (uniformly hyperbolic systems in [29], expanding Thurston maps in [34], symbolic dynamics in [28], the C^r category in [20], and joint optimization allowing variation of the system in [25, 26]). The question of *analytic positive entropy*, whether a positive entropy measure can be the unique maximizing measure for a real analytic function [30, Problem 3.12], was answered negatively by Gao and Shen [19]: for a real analytic expanding circle map, every maximizing measure of a real analytic function has zero entropy, unless the function is cohomologous to a constant (this also settles [30, Problem 3.11] negatively). The present article concerns the remaining question of *explicit exposure*: Problem 3.9 above, together with its generalisation [30, Problem 4.4], which asks, for a transitive hyperbolic map with local product structure, for an explicit continuous function with a unique maximizing measure of full support.

For the doubling map, any continuous f whose unique maximizing measure is Lebesgue measure m is necessarily rather irregular: f cannot be Hölder continuous, nor, more generally, Dini continuous (i.e., admit a modulus of continuity ω_f that satisfies $\int_0^{1/2} \omega_f(s) \frac{ds}{s} < +\infty$), nor can it belong to the class of Walters functions (cf. [6, 50]). This is because, for functions f in these regularity classes, it is known to be possible to modify f , by adding a continuous coboundary (which has zero integral for every invariant measure), so that the resulting function attains its maximum at every point of the support of every f -maximizing measure (see e.g. [5, 6, 7, 12, 13, 36, 43]). Consequently, if a measure of full support such as m were f -maximizing, then every invariant measure would be f -maximizing, precluding uniqueness. This obstruction, and the results described above in the real analytic, Lipschitz, and C^r categories, are instances of a common phenomenon: regularity of the function constrains the complexity (and in particular the entropy) of its maximizing measures. By contrast, Theorems A, B, C, and D below construct, explicitly, continuous functions whose unique maximizing measure is Haar measure m_G (which has full support and maximal entropy, cf. [9]) for every continuous endomorphism T of a compact metrizable abelian group G such that m_G is ergodic. In particular, the case $d = 2$ of Theorem A solves [30, Problem 3.9], and Theorems A–D answer [30, Problem 4.4] for expanding circle endomorphisms, hyperbolic toral automorphisms, and full shifts.

1.2. Setting and main results. Throughout the article, G denotes a nontrivial compact metrizable abelian group, $T: G \rightarrow G$ a continuous endomorphism, and m_G normalised Haar measure on G , which we assume to be ergodic for T . We write $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

¹For example, the problem was discussed at the 2004 CIRM École Pluri-thématique de Théorie Ergodique that led to [30], and at problem sessions during the 2006 BIRS workshop on Measurable Dynamics: Theory & Applications (<https://www.birs.ca/workshops/2006/06w5079/report06w5079.pdf>), and the 2011 BIRS workshop on Ergodic Optimization (<https://www.birs.ca/workshops/2011/11w5039/report11w5039.pdf>).

for the circle group, identified with $\{z \in \mathbb{C} : |z| = 1\}$ in the usual way, and $m_{\mathbb{T}}$ for normalised Haar–Lebesgue measure on $\mathbb{T}$; for H a compact abelian group, $\widehat{H}$ denotes its dual (character) group. We write $\Gamma := \widehat{G}$, and $A := \widehat{T}$ for the dual endomorphism of Γ , given by $A\gamma := \gamma \circ T$. Further notation and basic facts will be given in Section 2.

For $f \in C(G, \mathbb{R})$, the quantity

$$Q(T, f) := \sup \left\{ \int f \, d\mu : \mu \in \mathcal{M}(G, T) \right\}$$

is called the *maximum ergodic average* of f , where $\mathcal{M}(G, T)$ denotes the space of T -invariant Borel probability measures on G . A measure $\mu \in \mathcal{M}(G, T)$ is called *f-maximizing* if $\int f \, d\mu = Q(T, f)$, and the collection of all f -maximizing measures is denoted by $\mathcal{M}_{\max}(f)$.

Ergodic optimization studies $\mathcal{M}_{\max}(f)$ for a prescribed function f . Here we consider the inverse problem: given $\mu \in \mathcal{M}(G, T)$, construct $f \in C(G, \mathbb{R})$ such that

$$(1.1) \quad \mathcal{M}_{\max}(f) = \{\mu\}.$$

In this case, we say that f *exposes* μ , or that f is an *exposing function* for μ , since (1.1) asserts that μ is an exposed point (cf. [1, p. 119]) of the convex set $\mathcal{M}(G, T)$, in view of the fact that each $f \in C(G, \mathbb{R})$ induces a continuous linear functional $\nu \mapsto \int f \, d\nu$ on the space of signed Borel measures on G endowed with the weak* topology.

We first record three classical settings, namely expanding circle endomorphisms, ergodic toral endomorphisms, and the one-sided full shift, in which the exposing function can be given by a Fourier (or Walsh) expansion, uniformly convergent in the classical frequency order, with every coefficient rational and determined exactly by a finite procedure. These are special cases of the general recursive construction of Theorem D below, whose defining sequence can, after relocating frequencies, be reassembled in this way; see Section 5. In fact, for every G and every enumeration of $\widehat{G} \setminus \{1\}$, the same procedure gives an exposing function whose Fourier series converges uniformly when summed in that order (Proposition 5.3); by Theorem E below, it never converges absolutely. We begin with expanding circle endomorphisms, the doubling map of [30, Problem 3.9] being the case $d = 2$.

Theorem A (Fourier exposure for expanding circle endomorphisms). *For an integer d with $|d| \geq 2$, let $T: \mathbb{T} \rightarrow \mathbb{T}$ be given by $T(x) = dx \pmod{1}$. Then there exists a function $\widetilde{F} \in C(\mathbb{T}, \mathbb{R})$ of the form*

$$\widetilde{F}(x) = \sum_{k=1}^{+\infty} c_k \cos(2\pi kx), \quad c_k \in \mathbb{Q},$$

such that the Fourier partial sums converge uniformly to $\widetilde{F}$ on $\mathbb{T}$, and Lebesgue measure is the unique T -invariant maximizing measure for $\widetilde{F}$. Moreover, each coefficient c_k is determined exactly by a finite procedure.

The second setting consists of the toral endomorphisms $T_B(x) = Bx \pmod{\mathbb{Z}^m}$ given by an $m \times m$ integer matrix B with $\det B \neq 0$ for which Lebesgue measure is ergodic, i.e., for which no eigenvalue of B is a root of unity (cf. [24, 42, 45]). For these, Theorem B is a higher-dimensional analogue of Theorem A.

For Fourier series on $\mathbb{T}^m := (\mathbb{R}/\mathbb{Z})^m$, we use the standard square partial sums; see [21, Definition 3.2.3, p. 184]. Namely, for $F(x) := \sum_{k \in \mathbb{Z}^m} c_k \exp(2\pi i \langle k, x \rangle)$ and $R \in \mathbb{N}_0$, the R -th square partial sum is

$$(1.2) \quad \Sigma_R F(x) := \sum_{\|k\|_{\infty} \leq R} c_k \exp(2\pi i \langle k, x \rangle).$$

Theorem B (Fourier exposure for ergodic toral endomorphisms). *For an integer $m \geq 2$, suppose that B is an $m \times m$ integer matrix with $\det B \neq 0$ such that the toral endomorphism $T_B: \mathbb{T}^m \rightarrow \mathbb{T}^m$ given by $T_B(x) = Bx \pmod{\mathbb{Z}^m}$ is ergodic with respect to Lebesgue measure. Then there exists a function $\tilde{F} \in C(\mathbb{T}^m, \mathbb{R})$ of the form*

$$\tilde{F}(x) = \sum_{k \in \mathbb{Z}^m \setminus \{0\}} c_k \exp(2\pi i \langle k, x \rangle), \quad c_k \in \mathbb{Q},$$

such that the square partial sums of the Fourier series converge uniformly to $\tilde{F}$ on $\mathbb{T}^m$, and Lebesgue measure is the unique T_B -invariant maximizing measure for $\tilde{F}$. Moreover, each coefficient c_k is determined exactly by a finite procedure.

The construction applies equally to symbolic dynamics, for any alphabet $\mathbb{Z}/q\mathbb{Z}$ with an integer $q \geq 2$ and for either one-sided or two-sided shifts; we present here the case $X := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}_0}$, i.e., the one-sided full shift on two symbols. We use the standard Walsh ordering of $\hat{X}$; see [18, p. 99]. For $k \in \mathbb{N}_0$, write

$$(1.3) \quad k = \sum_{j=0}^{+\infty} \varepsilon_j(k) 2^j, \quad \varepsilon_j(k) \in \{0, 1\}.$$

For $x = (x_j)_{j \in \mathbb{N}_0} \in X$, the k -th Walsh function is defined by $w_k(x) := (-1)^{\sum_{j=0}^{+\infty} \varepsilon_j(k) x_j}$. Then $\hat{X} = \{w_k : k \in \mathbb{N}_0\}$ and $w_0 = 1$. The functions w_k originate in Walsh's system of orthonormal functions on $[0, 1]$ [49], introduced as a uniformly bounded alternative to the earlier orthonormal system of Haar [22]²; their identification with the characters of X , in the form used here, goes back to Fine [17].

Theorem C (Fourier exposure for the one-sided full shift). *Let $T: X \rightarrow X$ be the one-sided shift on $X := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}_0}$. Then there exists a function $\tilde{F} \in C(X, \mathbb{R})$ of the form*

$$\tilde{F}(x) = \sum_{k=1}^{+\infty} c_k w_k(x), \quad c_k \in \mathbb{Q},$$

such that the partial sums of the Walsh–Fourier series converge uniformly to $\tilde{F}$ on X , and the $(1/2, 1/2)$ -Bernoulli measure is the unique T -invariant maximizing measure for $\tilde{F}$. Moreover, each coefficient c_k is determined exactly by a finite procedure.

Remark 1.1. The coefficients in Theorems A, B, and C have a property different from, and in a precise sense stronger than, the computability of $\tilde{F}$. To say that c_k is *determined exactly by a finite procedure* is to say that there is an algorithm which, given k , returns the *exact* value of c_k after finitely many operations in the sense of Remark 1.2 below, rather than an approximation to it. Such a statement is meaningful only because each c_k is rational; no analogue is to be expected for the values $\tilde{F}(x)$, which are limits of infinite series and can in general only be approximated. Exactness of the c_k , together with the uniform convergence of the partial sums in the prescribed order, implies in particular that $\tilde{F}$ is computable in the sense of Remark 1.2 below: the series is summed directly, with every term exact and with an explicit bound on the truncation error at each stage (see the proof of Proposition 5.3).

The three theorems above are in fact special cases, obtained by relocating frequencies, of a single construction that works for an arbitrary compact metrizable abelian group:

²Note that Haar measure itself, the subject of this article, was introduced some two decades later, in [23].

Theorem D (Uniformly convergent trigonometric exposure of Haar measure). *Let G be a nontrivial compact metrizable abelian group and $T: G \rightarrow G$ a continuous endomorphism for which the normalised Haar measure m_G is ergodic. There is a sequence $\{F_n\}_{n \in \mathbb{N}_0}$ of real-valued trigonometric polynomials on G with rational Fourier coefficients, produced by an explicit recursion whose parameters are given by explicit formulas depending only on the step index n , such that F_n converges, as $n \rightarrow +\infty$, uniformly to a function $F \in C(G, \mathbb{R})$ for which m_G is the unique T -invariant maximizing measure. More precisely,*

$$\int F dm_G = 0 \quad \text{and} \quad \int F d\nu < 0 \quad \text{for all } \nu \in \mathcal{M}(G, T) \setminus \{m_G\}.$$

The parameters of the recursion are the same for every G and T ; only the characters from which the trigonometric polynomials F_n are built depend on G and T . The construction is also quantitative: Proposition 4.4 gives an explicit lower bound for $\int F dm_G - \int F d\nu$ in terms of the separation function $D(\nu)$ of Definition 2.2.

Remark 1.2. We use *computable* in the sense of computable analysis (see e.g. [40]). Informally, a real number is computable if an algorithm can produce a rational approximation to it within any prescribed error, and a function is computable if a single algorithm does this uniformly in the input and the requested accuracy, using sufficiently accurate approximations of the input.

For an abstract compact metrizable abelian group, computability is not an intrinsic notion: it can only be discussed after fixing a concrete effective representation of the group and the objects involved. Statements about computability should therefore be understood relative to a fixed presentation of G , its dual group Γ , and the dual endomorphism A .

Relative to such a presentation, we say that a quantity is produced by finitely many *operations* if it is obtained from the input by finitely many applications of the following, each of which is exact:

- (O1) the field operations of $\mathbb{Q}$, and the ceiling of a rational;
- (O2) multiplication and inversion in Γ , application of $A = \hat{T}$ to a character, and the test of equality of two characters;
- (O3) passage from $r \in \mathbb{N}$ to the character γ_r of the enumeration (2.3);
- (O4) formation of $P + Q$, PQ , $P \circ T^j$, and $R \circ P$, for $P, Q \in \mathcal{F}_{\mathbb{Q}}$ (the real-valued trigonometric polynomials with rational Fourier coefficients; see Section 2), $j \in \mathbb{N}_0$, and $R \in \mathbb{Q}[t]$, carried out on the finite lists of coefficients representing the arguments (Lemma 2.4);
- (O5) Haar integration of an element of $\mathcal{F}_{\mathbb{Q}}$, which returns the coefficient of the trivial character.

The recursion of Subsection 4.2 uses finitely many of these operations at each step.

Consequently, given $N \in \mathbb{N}$, an algorithm produces the complete finite Fourier expansion of F_N , and F_N can be evaluated at any point to any prescribed accuracy by a finite computation. Moreover, Proposition 4.3 gives the explicit bound $\|F - F_N\|_{\infty} < \frac{16}{105} \delta_N$. Since δ_N is explicitly defined, this bound gives a computable rate of convergence: from a target accuracy $\varepsilon > 0$, one can compute an N such that $\|F - F_N\|_{\infty} < \varepsilon/2$, and then evaluate F_N to within $\varepsilon/2$. It follows that F is computable in the above sense; see e.g. [40].

A function $f \in C(G, \mathbb{R})$ is called a *coboundary* if $f = u - u \circ T$ for some $u \in C(G, \mathbb{R})$; see e.g. [3, Section 1]. Every coboundary has zero integral with respect to every measure in $\mathcal{M}(G, T)$. In general, the space of coboundaries need not be closed in $C(G, \mathbb{R})$. Following [8, Section 3], an element of its uniform closure is called a *weak coboundary*. Equivalently, $f \in C(G, \mathbb{R})$ is a weak coboundary if and only if

$$\int f d\mu = 0 \quad \text{for all } \mu \in \mathcal{M}(G, T);$$

see [8, Lemma 3]. Thus every coboundary is a weak coboundary, although the converse need not hold.

Recall that a *character* of G is a continuous group homomorphism $\gamma: G \rightarrow \mathbb{T}$, and that the characters form the dual group $\Gamma = \widehat{G}$ under pointwise multiplication; see [18, Section 4.1], and Section 2 for further notation. For $f \in C(G, \mathbb{C})$ and $\gamma \in \widehat{G}$, its Fourier coefficient at γ is $\widehat{f}(\gamma) = \int f \bar{\gamma} dm_G$. We denote by $A(G)$ the *Wiener algebra* (or *Fourier algebra*, cf. [18, p. 132]) of G , defined as

$$A(G) := \left\{ f \in C(G, \mathbb{C}) : \sum_{\gamma \in \widehat{G}} |\widehat{f}(\gamma)| < +\infty \right\}.$$

Theorem E (Exposing functions cannot lie in the Wiener algebra). *Let G be a nontrivial compact metrizable abelian group and $T: G \rightarrow G$ a continuous endomorphism for which the normalised Haar measure m_G is ergodic. Suppose $F \in C(G, \mathbb{R})$ belongs to the Wiener algebra $A(G)$. If m_G is a T -invariant maximizing measure for F , then $F - Q(T, F)$ is a weak coboundary, and therefore every measure in $\mathcal{M}(G, T)$ is F -maximizing.*

Since G is nontrivial, $\mathcal{M}(G, T)$ is not a singleton (it contains $\delta_{e_G} \neq m_G$, as $T(e_G) = e_G$), so Theorem E shows that no continuous function whose unique maximizing measure is m_G lies in $A(G)$. In particular, the Fourier series in Theorems A, B, and C converge only conditionally: since $\gamma(e_G) = 1$ for every character γ , uniform convergence in every order of summation would give absolute convergence at e_G , and hence $\tilde{F} \in A(G)$. Theorem E is not a consequence of the regularity obstruction described in Subsection 1.1: it requires no expansion of T , and, for the doubling map, the Wiener algebra contains functions which are not Walters functions. Its conclusion is sharp, in that $F - Q(T, F)$ need not be a coboundary (Proposition 6.7).

Theorem A produces a uniformly convergent Fourier series with rational coefficients, each of which can be determined exactly by a finite procedure. It is natural to ask whether the coefficient sequence admits a closed form. We show that it does not: the sequence is not P-recursive, that is, it satisfies no nontrivial linear recurrence with polynomial coefficients; see Section 7 for the definition, and for the sense in which this excludes a closed form. This is a consequence of the following result, whose first statement concerns continuous functions on $\mathbb{T}$ alone and may be of independent interest. In its second statement, the self-map $T: \mathbb{T} \rightarrow \mathbb{T}$ need not be a group endomorphism.

Theorem F (Functions with P-recursive Fourier coefficients). *Let $F \in C(\mathbb{T}, \mathbb{R})$, and suppose that $\{\widehat{F}(n)\}_{n \in \mathbb{N}_0}$ is P-recursive, i.e., satisfies a nontrivial linear recurrence with polynomial coefficients. Then F is Hölder continuous, and real analytic outside a finite subset of $\mathbb{T}$. Moreover, if $T \in C^1(\mathbb{T}, \mathbb{T})$ satisfies $|T'| > 1$ on $\mathbb{T}$, and some F -maximizing measure for T has full support, then $F - Q(T, F)$ is a coboundary.*

The second statement in Theorem F follows from the first together with the Mañé lemma, by the argument recalled in Subsection 1.1. For $T(x) = dx \pmod{1}$ with $|d| \geq 2$, as in Theorem A, Lebesgue measure is $\tilde{F}$ -maximizing and has full support. The coefficients c_k of Theorem A are therefore not P-recursive: otherwise $\tilde{F} - Q(T, \tilde{F})$ would be a coboundary, and every invariant measure would be $\tilde{F}$ -maximizing, contradicting Theorem A. Equivalently, by Lemma 7.2 and Remark 7.3, the function $\tilde{F}$ satisfies no nontrivial linear differential equation with trigonometric-polynomial coefficients. More generally, if $T \in C^1(\mathbb{T}, \mathbb{T})$ satisfies $|T'| > 1$, then no continuous function whose unique T -invariant maximizing measure has full support has P-recursive Fourier coefficients, since such a T has more than one invariant probability measure.

1.3. Proof strategy. The set $\mathcal{M}(G, T) \setminus \{m_G\}$ is exhausted by the sets $\{\nu \in \mathcal{M}(G, T) : D(\nu) \geq \rho\}$, $\rho \downarrow 0$, where D is a weak* continuous gauge on $\mathcal{M}(G, T)$, built from the Fourier coefficients of ν , vanishing only at m_G (Definition 2.2 and Lemma 2.3). Each of these sets is weak* compact, so on each of them separation from m_G can be witnessed by a single trigonometric polynomial; the difficulty is to add together infinitely many such polynomials, at all scales $\rho \downarrow 0$ simultaneously, without the later ones destroying the gaps created by the earlier ones.

The construction alternates two procedures to overcome this. First, from finite character-orbit averages $B_{r,N}$ we build single-frequency detectors $K_{r,N}$ (Definition 2.5), and combine finitely many of these, following a truncation of D , into a trigonometric polynomial $\mathcal{S}_\rho$ which has zero Haar mean, is everywhere at most $\rho/4$, and has integral below $-\rho/4$ against every invariant measure ν with $D(\nu) \geq \rho$ (Proposition 2.7). Second, we flatten. A sufficiently long Birkhoff average of a trigonometric polynomial with zero Haar mean is cohomologous to the polynomial itself and is small in $L^2(m_G)$. Subtracting a Bernstein polynomial approximation of the positive part of this average, and adding back the Haar mean of that approximation, therefore lowers the maximum ergodic average below a prescribed positive threshold, while preserving the Haar mean and the rationality of the Fourier coefficients (Proposition 3.4). The parameters in Section 4 are chosen so that this does not destroy the negative gaps created at earlier steps. Alternating these two procedures at the scales $\rho_n = 2^{-n}$, with an explicit schedule of amplitudes, produces the sequence $\{F_n\}_{n \in \mathbb{N}_0}$ and proves Theorem D (Section 4).

To obtain the Fourier series of Theorems A, B, and C, Section 5 relocates the frequencies of the increments $F_n - F_{n-1}$: each increment is split into many relocated copies, so that no truncation of the resulting series can remove more than a prescribed amount from a single block of frequencies.

For Theorem E, the hypothesis $F \in A(G)$ allows $\int F d\nu$, for $\nu \in \mathcal{M}(G, T)$, to be written as a sum over the orbits of the dual endomorphism A , each orbit contributing one Fourier coefficient of ν multiplied by the sum of the Fourier coefficients of F over that orbit. If the sum over some nontrivial orbit is nonzero, we perturb m_G within $\mathcal{M}(G, T)$ so that, to first order, only the Fourier coefficients on that orbit and its inverse change, and the F -integral increases. The perturbation is obtained by averaging under T a Riesz product built on a dissociate subsequence of the orbit, and the first-order terms survive the averaging because the subsequence has positive density in the orbit: a theorem of Lind [35] allows the subsequence to be taken along an arithmetic progression of iterates of A .

For Theorem F, P-recursiveness of the Fourier coefficients translates into a distributional linear differential equation for F with trigonometric-polynomial coefficients, so F is real analytic away from the finitely many zeros of the leading coefficient. Near such a zero x_0 , which may be an irregular singular point, classical asymptotic theory [2, 15, 52] describes F by asymptotic expansions built from terms $e^{Q(r)} r^\nu (\log r)^k$, where $r := |x - x_0|$ and Q is a polynomial in a fractional power of $1/r$. The main step is to show that, in the expansion of $F(x) - F(x_0)$, continuity of F excludes every term that does not tend to 0. This is not automatic, since such terms may oscillate at different rates and might cancel one another; a mean-square argument shows that they cannot. The surviving terms are $\mathcal{O}(r^\beta)$ for some $\beta > 0$, and together with a bound on F' by a power of $1/r$, this gives Hölder continuity by interpolation.

1.4. Organisation. Section 2 recalls the relevant Fourier-analytic preliminaries, introduces the separation function D of Definition 2.2, and constructs from finite character-orbit averages the detectors $K_{r,N}$ and $\mathcal{S}_\rho$ of Proposition 2.7. Section 3 develops the flattening procedure of Proposition 3.4. Section 4 combines the two into the recursive construction of Subsection 4.2, proving Theorem D. Section 5 relocates frequencies, carrying this out in Subsection 5.2 explicitly for expanding circle endomorphisms, the one-sided full shift, and ergodic toral endomorphisms, thereby proving Theorems A–C. The final two sections are independent of the construction, and of each other, and demonstrate the extent to which the irregularity of the functions it produces is unavoidable. Section 6 proves Theorem E, and gives the example of Proposition 6.7. Section 7 proves Theorem F.

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2. DETECTORS AND QUANTITATIVE SEPARATION

Fourier coefficients separate invariant measures from Haar measure m_G . We first construct a detector for a single Fourier coefficient in Lemma 2.6 and then combine finitely many such detectors to obtain a quantitative gap at a prescribed scale in Proposition 2.7.

We begin by recalling some standard notation and facts from Fourier analysis on compact abelian groups; see [18].

Notation. We identify $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the circle group $\{z \in \mathbb{C} : |z| = 1\}$ in the usual way. A *character* of G is a continuous group homomorphism $\gamma : G \rightarrow \mathbb{T}$. We write $\Gamma := \widehat{G}$ for the group of all characters of G , with pointwise multiplication, so that for all $\gamma, \eta \in \Gamma$ and $x \in G$,

$$(\gamma\eta)(x) = \gamma(x)\eta(x) \quad \text{and} \quad \gamma^{-1}(x) = \overline{\gamma(x)}.$$

Let e_G denote the identity element of G , and let $1 \in \Gamma$ denote the trivial character, given by $1(x) = 1$ for all $x \in G$. Then $\gamma(e_G) = 1$ for all $\gamma \in \Gamma$. Recall that $A = \widehat{T}$ is given by $A\gamma := \gamma \circ T$. Since m_G is T -invariant and has full support, T is surjective, and hence A is injective. For a Borel probability measure ν on G and $\gamma \in \Gamma$, define the *Fourier coefficient* of ν at γ by $\widehat{\nu}(\gamma) := \int \overline{\gamma} d\nu$. If $\nu \in \mathcal{M}(G, T)$, then

$$(2.1) \quad \widehat{\nu}(A\gamma) = \widehat{\nu}(\gamma) \quad \text{for all } \gamma \in \Gamma.$$

The orthogonality of characters with respect to Haar measure is standard; see [18, Proposition 4.4]. In particular, for every nontrivial $\gamma \in \Gamma$,

$$(2.2) \quad \int \overline{\gamma} dm_G = 0.$$

We require (one direction of) the familiar criterion that m_G is T -ergodic if and only if A has no nontrivial finite orbit on Γ (cf. [24, 42, 45]; we include a proof for completeness).

Lemma 2.1 (Distinct characters along dual orbits). *The dual endomorphism $A = \widehat{T}$ is injective. Moreover, for every nontrivial $\gamma \in \Gamma$, the characters $\gamma, A\gamma, A^2\gamma, \dots$ are pairwise distinct.*

Proof. The injectivity of A follows from the surjectivity of T , as noted above. Suppose that $A^j\gamma = A^k\gamma$ for some $0 \leq j < k$. Since A , and hence A^j , is injective, $A^j(A^{k-j}\gamma) = A^j\gamma$ forces $A^{k-j}\gamma = \gamma$, so $A^p\gamma = \gamma$ for some least $p \geq 1$. Injectivity also gives $A^\ell\gamma \neq 1$ for every $\ell \geq 0$, since $A^\ell 1 = 1$ and $\gamma \neq 1$. The function $H := \sum_{\ell=0}^{p-1} A^\ell\gamma$ satisfies $H \circ T = H$, so by ergodicity H is constant m_G -almost everywhere. Since m_G has full support and H is continuous, H is a constant function. By (2.2), $\int H dm_G = 0$, so $H \equiv 0$, contradicting the fact that $H(e_G) = p > 0$. $\square$

Since G is compact metrizable, $L^2(G, m_G)$ is separable. Moreover, $\widehat{G}$ is an orthonormal basis of $L^2(G, m_G)$ by [18, Corollary 4.27]; hence $\widehat{G}$ is countable. Since G is nontrivial, $\Gamma \neq \{1\}$, and Lemma 2.1 then shows that Γ is infinite. Fix an enumeration

$$(2.3) \quad \Gamma \setminus \{1\} = \{\gamma_r : r \in \mathbb{N}\},$$

in which each nontrivial character occurs exactly once, and set $\gamma_0 := 1$.

Definition 2.2. Define the *separation function* $D : \mathcal{M}(G, T) \rightarrow [0, 1]$ by $D(\nu) := \sum_{r=1}^{+\infty} 2^{-r} |\widehat{\nu}(\gamma_r)|^2$, for $\nu \in \mathcal{M}(G, T)$.

We record the two properties of D that will be used.

Lemma 2.3 (Quadratic Fourier separation). *The separation function D is weak* continuous. Moreover, for $\nu \in \mathcal{M}(G, T)$, $D(\nu) = 0$ if and only if $\nu = m_G$. In particular, $D(\nu) > 0$ for every $\nu \in \mathcal{M}(G, T) \setminus \{m_G\}$.*

Proof. For each $r \in \mathbb{N}$, the map $\nu \mapsto \widehat{\nu}(\gamma_r)$ is weak* continuous. Since $2^{-r}|\widehat{\nu}(\gamma_r)|^2 \leq 2^{-r}$ for every $r \in \mathbb{N}$ and every probability measure ν , the series defining D converges uniformly. Hence D is weak* continuous. If $D(\nu) = 0$, then every nontrivial Fourier coefficient of ν vanishes, hence ν and m_G have the same Fourier coefficients. By uniqueness of Fourier coefficients for finite Borel measures on compact abelian groups (cf. [18, Corollary 4.34]), $\nu = m_G$. Conversely, $D(m_G) = 0$, since $\widehat{m_G}(\gamma_r) = 0$ for every $r \in \mathbb{N}$ by (2.2). $\square$

The function $D: \mathcal{M}(G, T) \rightarrow [0, 1]$ measures how far an invariant measure is from m_G , but it is not itself of the form $\nu \mapsto \int f d\nu$ with f continuous. We now convert it into a function of this form, beginning with the detector attached to a single Fourier coefficient. We first fix some notation.

Notation. A *trigonometric polynomial* on G is a function of the form $P = \sum_{\gamma \in E} a_\gamma \gamma$, where $E \subseteq \Gamma$ is finite and $a_\gamma \in \mathbb{C}$. The set of these trigonometric polynomials is denoted by $\mathcal{P}$. For $f \in C(G, \mathbb{C})$ and $\gamma \in \Gamma$, we define the *Fourier coefficient* of f at γ by $\widehat{f}(\gamma) := \int f \bar{\gamma} dm_G$, so that $\widehat{P}(\gamma) = a_\gamma$ for $\gamma \in E$ and $\widehat{P}(\gamma) = 0$ otherwise, and we set $\text{supp } \widehat{P} := \{\gamma \in \Gamma : \widehat{P}(\gamma) \neq 0\}$. Recall that the *Wiener algebra* $A(G)$ consists of those $f \in C(G, \mathbb{C})$ whose Fourier coefficients $(\widehat{f}(\gamma))_{\gamma \in \Gamma}$ lie in $\ell^1(\Gamma)$, equipped with the norm

$$(2.4) \quad \|f\|_A := \sum_{\gamma \in \Gamma} |\widehat{f}(\gamma)|;$$

it is a commutative Banach algebra under pointwise multiplication; moreover $\|f\|_\infty \leq \|f\|_A$ and $\|f_1 f_2\|_A \leq \|f_1\|_A \|f_2\|_A$ for $f, f_1, f_2 \in A(G)$. Every trigonometric polynomial lies in $A(G)$.

Let $\mathcal{F}_\mathbb{Q} = \mathcal{F}_\mathbb{Q}(G)$ denote the set of inversion-invariant trigonometric polynomials with rational Fourier coefficients; equivalently,

$$\mathcal{F}_\mathbb{Q}(G) = \{P \in \mathcal{P} : \widehat{P}(\gamma) \in \mathbb{Q} \text{ and } \widehat{P}(\gamma^{-1}) = \widehat{P}(\gamma) \text{ for each } \gamma \in \Gamma\},$$

that is, the real-valued trigonometric polynomials all of whose Fourier coefficients are rational. With respect to the fixed enumeration $\Gamma \setminus \{1\} = \{\gamma_r : r \in \mathbb{N}\}$ together with $\gamma_0 := 1$, we may equivalently write $P = \sum_{r=0}^{+\infty} a_r \gamma_r$, where only finitely many a_r are nonzero, and hence $\|P\|_A = \sum_{r=0}^{+\infty} |a_r|$.

The set $\mathcal{F}_\mathbb{Q}$ is closed under the operations used in the construction:

Lemma 2.4. *Suppose $P, Q \in \mathcal{F}_\mathbb{Q}(G)$, $j \in \mathbb{N}_0$, and $R \in \mathbb{Q}[t]$. Then $P + Q$, PQ , $P \circ T^j$, and $R \circ P$ all belong to $\mathcal{F}_\mathbb{Q}(G)$. Moreover,*

$$\text{supp } \widehat{P \circ T^j} = A^j(\text{supp } \widehat{P}) \quad \text{and} \quad \|P \circ T^j\|_A = \|P\|_A.$$

Proof. It follows directly from the definition, since characters are closed under multiplication and inversion. The support identity follows from $\gamma \circ T^j = A^j \gamma$ for all $\gamma \in \Gamma$. Since A is injective by Lemma 2.1, this identity shows moreover that $\gamma \mapsto \gamma \circ T^j$ is a bijection from $\text{supp } \widehat{P}$ onto $\text{supp } \widehat{P \circ T^j}$, so $\|P \circ T^j\|_A = \sum_{\gamma \in \text{supp } \widehat{P}} |\widehat{P}(\gamma)| = \|P\|_A$. $\square$

We now define the detector attached to the Fourier coefficient at γ_r , using averages of γ_r along its orbit under A .

Definition 2.5. Suppose $r, N \in \mathbb{N}$. For each $x \in G$, define $B_{r,N}(x) := \frac{1}{N} \sum_{j=0}^{N-1} \overline{A^j \gamma_r(x)}$, and define the real-valued function

$$K_{r,N}(x) := \frac{1}{N} - |B_{r,N}(x)|^2 = -\frac{1}{N^2} \sum_{0 \leq \ell < j < N} ((A^j \gamma_r)(x)(A^\ell \gamma_r)^{-1}(x) + (A^\ell \gamma_r)(x)(A^j \gamma_r)^{-1}(x)).$$

In particular, $K_{r,N} \in \mathcal{F}_\mathbb{Q}(G)$.

The detector $K_{r,N}$ has the following properties:

Lemma 2.6 (A real-valued Fourier detector). *Suppose $r, N \in \mathbb{N}$. Then the following hold:*

- (i) $\|K_{r,N}\|_\infty \leq 1$ and $K_{r,N} \leq \frac{1}{N}$.
- (ii) $\int K_{r,N} dm_G = 0$ and $\int K_{r,N} d\nu \leq \frac{1}{N} - |\widehat{\nu}(\gamma_r)|^2$ for all $\nu \in \mathcal{M}(G, T)$.
- (iii) $\|K_{r,N}\|_A < 1$.

Proof. (i) For each character $\gamma \in \Gamma$, $\|\gamma\|_\infty = 1$, thus $\|B_{r,N}\|_\infty \leq 1$. Hence, $\frac{1}{N} - 1 \leq K_{r,N}(x) \leq \frac{1}{N}$ for all $x \in G$. This proves (i).

(ii) By Lemma 2.1, the characters $A^j \gamma_r$ are distinct for $0 \leq j < N$. Thus their orthogonality gives $\int |B_{r,N}|^2 dm_G = \frac{1}{N}$, so $K_{r,N}$ has zero Haar mean. By (2.1), $\int B_{r,N} d\nu = \frac{1}{N} \sum_{j=0}^{N-1} \widehat{\nu}(A^j \gamma_r) = \widehat{\nu}(\gamma_r)$. Thus, by the Cauchy–Schwarz inequality, $\int |B_{r,N}|^2 d\nu \geq |\int B_{r,N} d\nu|^2 = |\widehat{\nu}(\gamma_r)|^2$, which gives $\int K_{r,N} d\nu \leq \frac{1}{N} - |\widehat{\nu}(\gamma_r)|^2$.

(iii) By Definition 2.5, $\|K_{r,N}\|_A \leq (N-1)/N < 1$. $\square$

At each scale ρ , the following function $\mathcal{S}_\rho$ is designed to separate m_G from all measures ν satisfying $D(\nu) \geq \rho$.

Notation. For $0 < \rho \leq 1$, set

$$(2.5) \quad R(\rho) := \lceil \log_2(2/\rho) \rceil, \quad N(\rho) := \lceil 4/\rho \rceil,$$

and define

$$(2.6) \quad \mathcal{S}_\rho := \sum_{r=1}^{R(\rho)} 2^{-r} K_{r,N(\rho)}.$$

It follows directly from Lemma 2.4 that $\mathcal{S}_\rho \in \mathcal{F}_{\mathbb{Q}}$.

Proposition 2.7 (Inversion-invariant trigonometric polynomial separation). *Suppose $\rho \in (0, 1]$. Then the following hold:*

- (i) $\int \mathcal{S}_\rho dm_G = 0$, $\|\mathcal{S}_\rho\|_\infty < 1$, and $\|\mathcal{S}_\rho\|_A < 1$.
- (ii) For each $x \in G$, $\mathcal{S}_\rho(x) < \rho/4$.
- (iii) For each $\nu \in \mathcal{M}(G, T)$ with $D(\nu) \geq \rho$, we have $\int \mathcal{S}_\rho d\nu < -\rho/4$.

Proof. (i) By Lemma 2.6, for all $r, N \in \mathbb{N}$, $\|K_{r,N}\|_\infty \leq 1$, $\int K_{r,N} dm_G = 0$, and $\|K_{r,N}\|_A < 1$. Thus $\int \mathcal{S}_\rho dm_G = 0$, $\|\mathcal{S}_\rho\|_\infty \leq \sum_{r=1}^{R(\rho)} 2^{-r} < 1$, and $\|\mathcal{S}_\rho\|_A < \sum_{r=1}^{R(\rho)} 2^{-r} < 1$.

(ii) It follows from Lemma 2.6 (i) and (2.5) that $K_{r,N(\rho)} \leq N(\rho)^{-1} \leq \rho/4$. Then by (2.6), we see that for each $x \in G$, $\mathcal{S}_\rho(x) \leq \frac{\rho}{4} \sum_{r=1}^{R(\rho)} 2^{-r} < \rho/4$.

(iii) Let $\nu \in \mathcal{M}(G, T)$ satisfy $D(\nu) \geq \rho$. Then $\sum_{r=1}^{R(\rho)} 2^{-r} |\widehat{\nu}(\gamma_r)|^2 \geq \rho - 2^{-R(\rho)} \geq \frac{\rho}{2}$. Combining this with Lemma 2.6 (ii), we get $\int \mathcal{S}_\rho d\nu \leq \sum_{r=1}^{R(\rho)} 2^{-r} (N(\rho)^{-1} - |\widehat{\nu}(\gamma_r)|^2) < \frac{\rho}{4} - \frac{\rho}{2} = -\frac{\rho}{4}$. $\square$

3. QUANTITATIVE AVERAGING AND POSITIVE-PART FLATTENING

The detectors $\mathcal{S}_\rho$, $0 < \rho \leq 1$, constructed in Proposition 2.7 separate Haar measure from every invariant measure at scale ρ . Since no single scale detects every invariant measure other than m_G , the construction must combine detectors from infinitely many scales, and the difficulty in doing so is to preserve the gaps already created by earlier detectors: at each stage, the newly added detector must contribute a definite negative amount to the measures it detects, while whatever positive amount it contributes to the remaining invariant measures must be kept under control.

To control this positive contribution, we combine two estimates: an L^2 bound for finite Birkhoff averages, together with the fact that a finite Birkhoff average is cohomologous to the original function (Lemmas 3.1 and 3.2); and a Bernstein polynomial approximation of the positive-part function, with

an explicit error bound (Lemma 3.3). Together these results underlie Proposition 3.4, which supplies the flattening procedure used in the recursion.

Notation. Throughout, $\|\cdot\|_{L^2}$ denotes the $L^2(m_G)$ -norm. For $M \in \mathbb{N}$, $H \in C(G, \mathbb{R})$, and $x \in G$, define the M -th Birkhoff sum by $S_M H(x) := \sum_{j=0}^{M-1} H(T^j(x))$.

We begin with an L^2 bound for finite Birkhoff averages.

Lemma 3.1 (A finite L^2 estimate). *Suppose $H \in \mathcal{F}_{\mathbb{Q}}$ satisfies $\int H dm_G = 0$. Then for each $M \in \mathbb{N}$,*

$$\|M^{-1}S_M H\|_{L^2} \leq M^{-1/2}\|H\|_A.$$

Proof. Suppose $H = \sum_{r=1}^{+\infty} a_r \gamma_r$, where only finitely many a_r are nonzero. Then

$$\left\| \frac{S_M H}{M} \right\|_{L^2} = \left\| \sum_{r=1}^{+\infty} a_r \left(\frac{\sum_{k=0}^{M-1} A^k \gamma_r}{M} \right) \right\|_{L^2} \leq \sum_{r=1}^{+\infty} |a_r| \left\| \frac{\sum_{k=0}^{M-1} A^k \gamma_r}{M} \right\|_{L^2} \leq \frac{\|H\|_A}{M^{1/2}},$$

where the final inequality follows from Lemma 2.1, which makes $\gamma_r, A\gamma_r, \dots, A^{M-1}\gamma_r$ pairwise distinct, and hence orthonormal in $L^2(m_G)$ by (2.2). $\square$

The finite L^2 estimate above controls the finite Birkhoff average $M^{-1}S_M H$. We shall also use the standard fact that a finite Birkhoff average is cohomologous to the original function; see, for example, [3, Section 1]. For $H \in C(G, \mathbb{R})$ and $M \in \mathbb{N}$, define

$$U_{M,H} := \frac{1}{M} \sum_{j=0}^{M-2} (M-1-j) H \circ T^j,$$

with $U_{1,H} := 0$.

Lemma 3.2 (A finite coboundary identity). *For $H \in C(G, \mathbb{R})$ and $M \in \mathbb{N}$, the following hold:*

- (i) $H - M^{-1}S_M H = U_{M,H} - U_{M,H} \circ T$.
- (ii) If $H \in \mathcal{F}_{\mathbb{Q}}$, then $U_{M,H} \in \mathcal{F}_{\mathbb{Q}}$ and $\|U_{M,H}\|_{\infty} \leq \|H\|_A(M-1)/2$.

Proof. (i) This follows directly from the definition.

(ii) If $H \in \mathcal{F}_{\mathbb{Q}}$, then by Lemma 2.4, $U_{M,H} \in \mathcal{F}_{\mathbb{Q}}$, and the inequality comes from

$$\|U_{M,H}\|_{\infty} \leq \frac{1}{M} \sum_{j=0}^{M-2} (M-1-j) \|H\|_{\infty} = \frac{M-1}{2} \|H\|_{\infty} \leq \frac{M-1}{2} \|H\|_A. \quad \square$$

To preserve $\mathcal{F}_{\mathbb{Q}}$ throughout the construction, we approximate $t_+ := \max\{t, 0\}$ by a Bernstein polynomial. Let $d \in \mathbb{N}$ and $U > 0$. Set $y_j := U \left(\frac{2j}{d} - 1 \right)$ for $0 \leq j \leq d$. For $t \in \mathbb{R}$, set $u := (t + U)/(2U)$, and define

$$(3.1) \quad \mathcal{B}_{d,U}(t) := \sum_{j=0}^d y_j \binom{d}{j} u^j (1-u)^{d-j}.$$

Then on $[-U, U]$ (affinely identified with $[0, 1]$), $\mathcal{B}_{d,U}$ is the Bernstein polynomial of the positive-part function. Since $x \mapsto x_+$ is convex and 1-Lipschitz, the following standard estimate follows from [39, Theorem 7.1.8] and [37, Theorem 1]. We include the proof for the convenience of the reader.

Lemma 3.3 (Bernstein approximation of the positive part). *Suppose $U > 0$ and $d \in \mathbb{N}$. Then for $t \in [-U, U]$, $0 \leq \mathcal{B}_{d,U}(t) - t_+ \leq U/\sqrt{d}$. Moreover, if $U \in \mathbb{Q}$, then $\mathcal{B}_{d,U} \in \mathbb{Q}[t]$.*

Proof. Consider a binomial random variable X with distribution given by $\mathbb{P}(X = j) = \binom{d}{j} u^j (1-u)^{d-j}$ for integers $0 \leq j \leq d$. Set $Y := U(\frac{2X}{d} - 1)$. For $t \in [-U, U]$ we have $\mathcal{B}_{d,U}(t) = \mathbb{E}[Y_+]$, $\mathbb{E}[Y] = t$, and $\text{Var}(Y) = 4U^2 u(1-u)/d \leq U^2/d$. Since $x \mapsto x_+$ is convex, Jensen's inequality gives $\mathcal{B}_{d,U}(t) - t_+ = \mathbb{E}[Y_+] - (\mathbb{E}[Y])_+ \geq 0$; since $x \mapsto x_+$ is 1-Lipschitz, $\mathbb{E}[Y_+] - (\mathbb{E}[Y])_+ \leq \mathbb{E}[|Y - \mathbb{E}[Y]|]$, and the Cauchy-Schwarz inequality then gives $\mathbb{E}[|Y - \mathbb{E}[Y]|] \leq \sqrt{\text{Var}(Y)} \leq U/\sqrt{d}$. Finally, if $U \in \mathbb{Q}$, then $\mathcal{B}_{d,U} \in \mathbb{Q}[t]$ follows directly from the definition. $\square$

Combining the preceding three lemmas yields the flattening estimate used below.

Proposition 3.4 (Positive-part flattening). *Let $\theta, \eta > 0$, and $B \in \mathbb{Q}_{>0}$, and let $H \in \mathcal{F}_{\mathbb{Q}}$ satisfy $\int H dm_G = 0$ and $\|H\|_A \leq B$. Suppose $M \in \mathbb{N}$ satisfies*

$$(3.2) \quad \sup_G M^{-1} S_M H < \theta \quad \text{and} \quad M^{-1/2} B < \eta/2.$$

Set $d := 1 + \lceil (2B/\eta)^2 \rceil$, and define

$$(3.3) \quad P_M := M^{-1} S_M H, \quad p_M := \int \mathcal{B}_{d,B}(P_M) dm_G, \quad Q_M := p_M - \mathcal{B}_{d,B}(P_M).$$

Then the following hold:

- (i) $0 \leq p_M < \eta$.
- (ii) $\|Q_M\|_{\infty} < \theta + \eta$ and, for each $K \in \mathbb{N}$,

$$\sup_G K^{-1} S_K(H + Q_M) < \eta + K^{-1}(M-1)B.$$

In particular, $Q(T, H + Q_M) < \eta$.

- (iii) $H + Q_M \in \mathcal{F}_{\mathbb{Q}}$, $\int (H + Q_M) dm_G = 0$, and $\|H + Q_M\|_A < (1 + 2^d)B + \eta$.

Proof. (i) Since $\|P_M\|_A \leq \|H\|_A \leq B$, we have $\|P_M\|_{\infty} \leq B$. Lemma 3.3 gives $0 \leq \mathcal{B}_{d,B}(P_M) - (P_M)_+ \leq B/\sqrt{d} < \eta/2$. Thus $p_M \geq 0$ and

$$p_M = \int \mathcal{B}_{d,B}(P_M) dm_G \leq \int \left(|P_M| + \frac{\eta}{2} \right) dm_G \leq \|P_M\|_{L^2} + \frac{\eta}{2} < \eta,$$

where the last inequality follows from Lemma 3.1 together with (3.2).

(ii) By (3.3) and (3.2), $P_M < \theta$. Thus by Lemma 3.3, $\mathcal{B}_{d,B}(P_M) \in [0, \theta + \eta/2)$. Since $p_M \in [0, \eta)$ by (i), we have $-(\theta + \eta/2) < Q_M < \eta$ and thus $\|Q_M\|_{\infty} < \theta + \eta$. By Lemmas 3.2 and 3.3,

$$H + Q_M = p_M + (P_M - \mathcal{B}_{d,B}(P_M)) + U_{M,H} - U_{M,H} \circ T \leq p_M + U_{M,H} - U_{M,H} \circ T.$$

Hence, for every $K \in \mathbb{N}$,

$$\frac{S_K(H + Q_M)}{K} \leq p_M + \frac{U_{M,H} - U_{M,H} \circ T^K}{K} \leq p_M + \frac{2\|U_{M,H}\|_{\infty}}{K} < \eta + \frac{(M-1)B}{K}.$$

Suppose $\nu \in \mathcal{M}(G, T)$. By T -invariance, $\int (H + Q_M) d\nu \leq \int (p_M + U_{M,H} - U_{M,H} \circ T) d\nu = p_M$. Hence $Q(T, H + Q_M) \leq p_M < \eta$.

(iii) Since $B \in \mathbb{Q}$, Lemma 3.3 shows that $\mathcal{B}_{d,B} \in \mathbb{Q}[t]$. By Lemma 2.4, $\mathcal{B}_{d,B}(P_M) \in \mathcal{F}_{\mathbb{Q}}$. Its Haar integral p_M is therefore rational, being its Fourier coefficient at the trivial character. Consequently, $H + Q_M = H + p_M - \mathcal{B}_{d,B}(P_M) \in \mathcal{F}_{\mathbb{Q}}$. By (3.3) and $\int H dm_G = 0$, we have $\int (H + Q_M) dm_G = 0$.

Set $V := (P_M + B)/(2B)$. Since $\|P_M\|_A \leq \|H\|_A \leq B$, we have $\|V\|_A \leq 1$ and $\|1 - V\|_A \leq 1$. Hence, by (3.1) and (2.4), $\|\mathcal{B}_{d,B}(P_M)\|_A \leq \sum_{j=0}^d y_{j+} \binom{d}{j} \|V\|_A^j \|1 - V\|_A^{d-j} \leq B \sum_{j=0}^d \binom{d}{j} = 2^d B$. Therefore, $\|H + Q_M\|_A \leq \|H\|_A + p_M + \|\mathcal{B}_{d,B}(P_M)\|_A < B + \eta + 2^d B = (1 + 2^d)B + \eta$. $\square$

4. THE EXPLICIT CONSTRUCTION

In this section, we construct the sequence $\{F_n\}_{n \in \mathbb{N}_0}$ using Propositions 2.7 and 3.4. At each step, we first add a detector to enlarge the class of invariant measures on which a negative gap is obtained, and then apply the flattening procedure to control the positive contribution introduced at the new step without destroying the gaps obtained previously.

4.1. Scales and parameter choices. Suppose $n \in \mathbb{N}$. We set $\rho_n := 2^{-n}$ as the detection scale and consider $\mathcal{D}_n := \mathcal{S}_{\rho_n} = \sum_{r=1}^{n+1} 2^{-r} K_{r, 2^{n+2}}$ (cf. (2.6)). Recall from Proposition 2.7 that

$$(4.1) \quad \int \mathcal{D}_n dm_G = 0, \quad \|\mathcal{D}_n\|_\infty < 1, \quad \|\mathcal{D}_n\|_A < 1, \quad \sup_{x \in G} \mathcal{D}_n(x) < \frac{\rho_n}{4},$$

and that for each $\nu \in \mathcal{M}(G, T)$ satisfying $D(\nu) \geq \rho_n$,

$$(4.2) \quad \int \mathcal{D}_n d\nu < -\frac{\rho_n}{4}.$$

As $n \rightarrow +\infty$, the detectors $\mathcal{D}_n$ separate m_G from the nested sets $\{\nu \in \mathcal{M}(G, T) : D(\nu) \geq \rho_n\}$, whose union is $\mathcal{M}(G, T) \setminus \{m_G\}$. At step n , we add the scaled detector $a_n \mathcal{D}_n$, while the choice of a_n ensures that the new perturbation does not destroy the gaps created at previous steps.

Starting with $a_1 := \frac{1}{4}$ and $\delta_1 := a_1 \rho_1$, we recursively define $a_{n+1} := \delta_n/8$ and $\delta_{n+1} := a_{n+1} \rho_{n+1}$ for each $n \in \mathbb{N}$. Explicitly, for each $n \in \mathbb{N}$, we have

$$(4.3) \quad a_n = 2^{-\frac{n(n-1)}{2}-3n+1} \quad \text{and} \quad \delta_n = 2^{-\frac{n(n-1)}{2}-4n+1}.$$

For each $n \in \mathbb{N}_0$, we also set

$$(4.4) \quad b_n := \delta_{n+1}/64 = 2^{-\frac{n(n+1)}{2}-4n-9}.$$

4.2. The inductive construction. Let F_0 be the constant zero function on G , and set $B_0 := 0$, $\Theta_0 := 0$. Then (4.5) and (4.6) below hold for $n = 1$: indeed $F_0 \in \mathcal{F}_\mathbb{Q}$, $\int F_0 dm_G = 0$, $Q(T, F_0) = 0 < b_0 = 2^{-9}$, $\|F_0\|_A = 0 = B_0$, and $\sup_G K^{-1} S_K F_0 = 0 < b_0$ for every $K \in \mathbb{N}$.

Suppose $(F_{n-1}, \Theta_{n-1}, B_{n-1})$ with $B_{n-1}, \Theta_{n-1} \in \mathbb{Q}$ has been constructed for some $n \in \mathbb{N}$ and satisfies

$$(4.5) \quad F_{n-1} \in \mathcal{F}_\mathbb{Q}, \quad \int F_{n-1} dm_G = 0, \quad Q(T, F_{n-1}) < b_{n-1}, \quad \|F_{n-1}\|_A \leq B_{n-1},$$

and, for every $K \in \mathbb{N}$,

$$(4.6) \quad \sup_G K^{-1} S_K F_{n-1} < b_{n-1} + K^{-1} \Theta_{n-1}.$$

We now construct (F_n, Θ_n, B_n) with $B_n, \Theta_n \in \mathbb{Q}$.

We first add the detector and form the intermediate function

$$(4.7) \quad H_n := F_{n-1} + a_n \mathcal{D}_n.$$

By (4.2), (4.5), and the definition (4.4) of b_{n-1} , we see that for each $\nu \in \mathcal{M}(G, T)$ satisfying $D(\nu) \geq \rho_n$,

$$(4.8) \quad \int H_n d\nu < Q(T, F_{n-1}) - \frac{\delta_n}{4} < -\frac{15}{64} \delta_n.$$

Set $\bar{B}_n := B_{n-1} + a_n \in \mathbb{Q}$. By (4.5), (4.7), (4.1), and (4.6), we have $\|H_n\|_A < \bar{B}_n$, and, for each $K \in \mathbb{N}$,

$$(4.9) \quad \sup_G K^{-1} S_K H_n < b_{n-1} + K^{-1} \Theta_{n-1} + (1/4) \delta_n.$$

To ensure that $Q(T, F_n)$ is sufficiently small relative to δ_{n+1} , we use the flattening procedure to construct F_n from H_n . Note that $\int H_n dm_G = 0$ by the induction hypothesis and (4.1). We now

apply Proposition 3.4 to H_n with $2b_{n-1} + \delta_n/4$ in place of θ , b_n in place of η , $\bar{B}_n$ in place of B , and M_n in place of M , where

$$(4.10) \quad M_n := 1 + \lceil \max\{\Theta_{n-1}/b_{n-1}, (2\bar{B}_n/b_n)^2\} \rceil.$$

It follows from (4.9) and (4.10) that $\sup_G M_n^{-1} S_{M_n} H_n < 2b_{n-1} + \delta_n/4$. Moreover, by (4.10), $M_n^{-1/2} \bar{B}_n < b_n/2$. We then set the Bernstein polynomial degree

$$(4.11) \quad d_n := 1 + \lceil (2\bar{B}_n/b_n)^2 \rceil,$$

and consider

$$C_n := \int \mathcal{B}_{d_n, \bar{B}_n} \left(\frac{S_{M_n} H_n}{M_n} \right) dm_G \quad \text{and} \quad Q_n := C_n - \mathcal{B}_{d_n, \bar{B}_n} \left(\frac{S_{M_n} H_n}{M_n} \right).$$

We define

$$(4.12) \quad F_n := H_n + Q_n = H_n + C_n - \mathcal{B}_{d_n, \bar{B}_n} (M_n^{-1} S_{M_n} H_n).$$

By Proposition 3.4, we see that $F_n \in \mathcal{F}_{\mathbb{Q}}$, $\int F_n dm_G = 0$, $C_n < b_n$, and $Q(T, F_n) < b_n$. Moreover,

$$(4.13) \quad \|Q_n\|_{\infty} < 2b_{n-1} + (1/4)\delta_n + b_n,$$

and, for each $K \in \mathbb{N}$, $\sup_G K^{-1} S_K F_n < b_n + K^{-1}(M_n - 1)\bar{B}_n$. Moreover, $\|F_n\|_A < (1 + 2^{d_n})\bar{B}_n + b_n$. Thus, we define

$$(4.14) \quad \begin{aligned} B_n &:= (1 + 2^{d_n})\bar{B}_n + b_n = (1 + 2^{d_n})(B_{n-1} + a_n) + b_n \in \mathbb{Q} \quad \text{and} \\ \Theta_n &:= (M_n - 1)\bar{B}_n = (M_n - 1)(B_{n-1} + a_n) \in \mathbb{Q}. \end{aligned}$$

This completes the inductive step.

4.3. Exposure. In this subsection, we prove Theorem D and record the properties of $\{F_n\}_{n \in \mathbb{N}_0}$ for later reference.

Proposition 4.1 (Properties of the constructed sequence). *Let $\{F_n\}_{n \in \mathbb{N}_0}$, $\{\Theta_n\}_{n \in \mathbb{N}_0}$, and $\{B_n\}_{n \in \mathbb{N}_0}$ be the sequences recursively defined in Subsection 4.2. Then for each $n \in \mathbb{N}$ and each $K \in \mathbb{N}$,*

$$F_n \in \mathcal{F}_{\mathbb{Q}}, \quad \int F_n dm_G = 0, \quad Q(T, F_n) < b_n, \quad \|F_n\|_A < B_n, \quad \text{and} \quad \sup_G \frac{S_K F_n}{K} < b_n + \frac{\Theta_n}{K}.$$

Proof. These are precisely the conclusions of the recursive construction in Subsection 4.2. $\square$

We first show that F_n has a negative integral, of definite size, against every invariant measure detected at scale ρ_n .

Proposition 4.2 (Exposure estimate). *Let F_n be as in Proposition 4.1. If $n \in \mathbb{N}$ and $\nu \in \mathcal{M}(G, T)$ satisfy $D(\nu) \geq 2^{-n}$, then $\int F_n d\nu < -\frac{7}{32}\delta_n$.*

Proof. It follows from (4.12), Proposition 3.4 (i), and (4.8) that $\int F_n d\nu \leq \int H_n d\nu + C_n < -\frac{15}{64}\delta_n + b_n < -\frac{7}{32}\delta_n$, where the last inequality follows from the fact that $b_n \leq \delta_n/2^{10}$ (cf. (4.4) and (4.3)). $\square$

Proposition 4.2 gives a negative gap at each fixed step. We next estimate the change at each subsequent step.

Proposition 4.3 (Increment estimate). *Let $\{a_n\}_{n \in \mathbb{N}}$ be the sequence defined by (4.3). Then for each $n \in \mathbb{N}$,*

$$\|F_n - F_{n-1}\|_{\infty} < (8/7)a_n.$$

Consequently, $\{F_n\}_{n \in \mathbb{N}_0}$ converges uniformly to a function $F \in C(G, \mathbb{R})$, and $\|F - F_N\|_{\infty} < \frac{16}{105}\delta_N$ for each $N \in \mathbb{N}$.

Proof. Recall that $\mathcal{D}_n = \mathcal{S}_{\rho_n}$ (cf. Subsection 4.1). It follows from (4.7), (4.12), and (4.13) that $\|F_n - F_{n-1} - a_n \mathcal{D}_n\|_\infty < 2b_{n-1} + \frac{\delta_n}{4} + b_n$. By (4.1), we see that $\|F_n - F_{n-1}\|_\infty < a_n + 2b_{n-1} + \delta_n/4 + b_n$. Moreover, by (4.4) and (4.3), $2b_{n-1} + \frac{\delta_n}{4} + b_n \leq (\frac{1}{32} + \frac{1}{4} + \frac{1}{2048})\delta_n < \frac{2}{7}\delta_n$. Consequently, $\|F_n - F_{n-1}\|_\infty < a_n + \frac{2}{7}\delta_n = a_n + \frac{2}{7} \cdot 2^{-n}a_n \leq \frac{8}{7}a_n$. Hence by (4.3), F_n converges uniformly to some $F \in C(G, \mathbb{R})$ as $n \rightarrow +\infty$. By (4.3), $a_{n+1}/a_n = 2^{-(n+3)} \leq \frac{1}{16}$. Hence $\|F - F_N\|_\infty < \frac{8}{7} \sum_{n=N+1}^{+\infty} a_n \leq \frac{8}{7} \cdot \frac{a_{N+1}}{1-1/16} \leq \frac{16}{105}\delta_N$. $\square$

Combining Propositions 4.2 and 4.3 gives a quantitative form of exposure for the limit F .

Proposition 4.4 (Quantitative exposure bound). *Let F be as in Proposition 4.3. Then*

$$\int F \, d\mathbf{m}_G = 0,$$

and for each $\nu \in \mathcal{M}(G, T) \setminus \{m_G\}$,

$$(4.15) \quad \int F \, d\mathbf{m}_G - \int F \, d\nu > 2^{-(n(\nu)+1)(n(\nu)+6)/2},$$

where $n(\nu) := \max\{1, \lceil \log_2 \frac{1}{D(\nu)} \rceil\}$.

Proof. It follows from Proposition 4.1 and the uniform convergence of $\{F_n\}_{n \in \mathbb{N}_0}$ that $\int F \, d\mathbf{m}_G = 0$. Suppose $\nu \in \mathcal{M}(G, T) \setminus \{m_G\}$. By Lemma 2.3, $D(\nu) > 0$, so $n(\nu)$ is a well-defined positive integer and $2^{-n(\nu)} \leq D(\nu)$. It then follows from Propositions 4.2 and 4.3 that

$$\int F \, d\nu \leq \int F_{n(\nu)} \, d\nu + \|F - F_{n(\nu)}\|_\infty < -\frac{7}{32}\delta_{n(\nu)} + \frac{16}{105}\delta_{n(\nu)} < -\frac{\delta_{n(\nu)}}{16}.$$

Note that by (4.3), for each $n \in \mathbb{N}$, $\delta_n/16 = 2^{-(n+1)(n+6)/2}$. Now (4.15) follows. $\square$

We can now combine Propositions 4.1, 4.3, and 4.4 in order to prove Theorem D:

Proof of Theorem D. By Proposition 4.1, each $F_n \in \mathcal{F}_{\mathbb{Q}}$; in particular F_n is a real-valued trigonometric polynomial on G with rational Fourier coefficients. By Proposition 4.3, $\{F_n\}_{n \in \mathbb{N}_0}$ converges uniformly to a function $F \in C(G, \mathbb{R})$. By Proposition 4.4,

$$\int F \, d\mathbf{m}_G = 0 \quad \text{and} \quad \int F \, d\nu < 0 \quad \text{for each } \nu \in \mathcal{M}(G, T) \setminus \{m_G\}.$$

Consequently $Q(T, F) = \sup_{\mu \in \mathcal{M}(G, T)} \int F \, d\mu = 0$, and this supremum is attained only at $\mu = m_G$; that is, $\mathcal{M}_{\max}(F) = \{m_G\}$, so m_G is the unique T -invariant maximizing measure for F . $\square$

5. A UNIFORMLY CONVERGENT FOURIER SERIES

Each finite approximant $F_n \in \mathcal{F}_{\mathbb{Q}}(G)$ has a finite Fourier expansion, and hence all of its Fourier coefficients are determined explicitly by finitely many steps of the construction. It is therefore natural to ask whether an exposing function can itself be represented by a Fourier series converging in a prescribed frequency order.

To carry this out, we use the enumeration (2.3), which so far has been arbitrary, as the frequency order. For every G and every enumeration, relocating the frequencies of the increments $F_n - F_{n-1}$ gives a function $\tilde{F}$ which exposes m_G and whose Fourier series converges uniformly in the order of the enumeration (Propositions 5.3 and 5.4). Under additional effectivity assumptions on the enumeration, each Fourier coefficient of $\tilde{F}$ is determined exactly by a finite procedure (Proposition 5.5). In Subsection 5.2 we specialise to expanding circle endomorphisms, ergodic toral endomorphisms, and the one-sided full shift, where the classical frequency orders satisfy these assumptions; this gives Theorems A, B, and C.

5.1. A Fourier series with exact Fourier coefficients. For all $P \in C(G, \mathbb{R})$ and $J \in \mathbb{N}_0$ we have $\|P \circ T^J\|_\infty = \|P\|_\infty$, since T^J is surjective, and, by the T -invariance of ν ,

$$(5.1) \quad \int P \circ T^J d\nu = \int P d\nu \quad \text{for all } \nu \in \mathcal{M}(G, T).$$

For each $n \in \mathbb{N}$, set $\Delta_n := F_n - F_{n-1}$ and $\alpha_n := \|\Delta_n\|_\infty$. By Lemma 2.4, $\Delta_n \in \mathcal{F}_\mathbb{Q}$. We estimate the size of the increment Δ_n . By Proposition 4.3, $\alpha_n < \frac{8}{7}a_n$, so by (4.3), we have

$$(5.2) \quad \sum_{n=1}^{+\infty} \alpha_n < \frac{8}{7} \sum_{n=1}^{+\infty} a_n < \frac{8}{7} \sum_{n=1}^{+\infty} 2^{-3n+1} < +\infty.$$

Since the increments are summable in the uniform norm, we may relocate them into separated frequency blocks: $\Delta_n \circ T^J$ has the same uniform norm as Δ_n and, by (5.1), the same integral with respect to each invariant measure, while, by Lemma 2.4, its frequency support is $A^J(\text{supp } \widehat{\Delta_n})$.

Relocation alone, however, does not suffice. A partial sum whose cutoff falls inside the frequency support of a single Δ_n may differ from Δ_n by as much as $\|\Delta_n\|_A$, and the Wiener norms $\|\Delta_n\|_A$ are in general far larger than the uniform norms α_n . We therefore also *split*: we replace Δ_n by K_n relocated copies of $K_n^{-1}\Delta_n$, placed in disjoint frequency blocks, where K_n is large enough that each block has Wiener norm at most 2^{-n} . The uniform norm of the sum does not increase, but no cutoff can now remove more than 2^{-n} from a single block. Concretely, for $n \in \mathbb{N}$ set

$$(5.3) \quad K_n := 1 + \lceil 2^n \|\Delta_n\|_A \rceil,$$

let $J_{n,s} \in \mathbb{N}_0$, $1 \leq s \leq K_n$, be integers to be specified below, and define

$$(5.4) \quad \tilde{\Delta}_{n,s} := \frac{1}{K_n} \Delta_n \circ T^{J_{n,s}} \quad \text{and} \quad \tilde{\Delta}_n := \sum_{s=1}^{K_n} \tilde{\Delta}_{n,s} = \frac{1}{K_n} \sum_{s=1}^{K_n} \Delta_n \circ T^{J_{n,s}}.$$

The *frequency support* of Δ_n is

$$(5.5) \quad E_n := \{\gamma \in \Gamma : \widehat{\Delta_n}(\gamma) \neq 0\},$$

which is finite by Proposition 4.1 and Lemma 2.4, and does not contain the trivial character because $\int \Delta_n dm_G = 0$. For convenience, we include the following fact.

Lemma 5.1 (Nonempty frequency support). *For each $n \in \mathbb{N}$, $\Delta_n(e_G) < 0$, and consequently, $E_n \neq \emptyset$.*

Proof. Since $T(e_G) = e_G$, the Dirac measure δ_{e_G} belongs to $\mathcal{M}(G, T)$. Moreover, $\widehat{\delta_{e_G}}(\gamma_r) = 1$ for each $r \in \mathbb{N}$, so Definition 2.2 gives $D(\delta_{e_G}) = 1 \geq \rho_n$. Hence (4.2) yields $\mathcal{D}_n(e_G) = \int \mathcal{D}_n d\delta_{e_G} < -\frac{\rho_n}{4}$. By Lemma 3.3, $\mathcal{B}_{d_n, \bar{B}_n} \geq 0$ on $[-\bar{B}_n, \bar{B}_n]$; since $\|M_n^{-1} S_{M_n} H_n\|_\infty \leq \|H_n\|_\infty \leq \|H_n\|_A < \bar{B}_n$ by Subsection 4.2, we have $M_n^{-1} S_{M_n} H_n(e_G) \in [-\bar{B}_n, \bar{B}_n]$, so (4.12) gives $Q_n(e_G) \leq C_n$, while Proposition 3.4 (i) gives $C_n < b_n$. Therefore, using (4.7), (4.12), and $\delta_n = a_n \rho_n$, we obtain

$$\Delta_n(e_G) = a_n \mathcal{D}_n(e_G) + Q_n(e_G) < -(1/4)a_n \rho_n + b_n = -(1/4)\delta_n + b_n < 0,$$

where the last inequality follows from (4.3) and (4.4). By Proposition 4.1, $\int \Delta_n dm_G = \int F_n dm_G - \int F_{n-1} dm_G = 0$. If $E_n = \emptyset$ then every Fourier coefficient of Δ_n vanishes, so $\Delta_n \equiv 0$, contradicting $\Delta_n(e_G) < 0$. $\square$

Now we index the relocated blocks $A^{J_{n,s}} E_n$ by the pairs (n, s) in lexicographic order.

Lemma 5.2 (Frequency relocation). *Let $E \subseteq \Gamma \setminus \{1\}$ be finite and $M \in \mathbb{N}_0$. Then there exists an integer $J \in [0, M|E|]$ such that*

$$(5.6) \quad A^J E \cap \{\gamma_1, \gamma_2, \dots, \gamma_M\} = \emptyset.$$

Equivalently, every character in $A^J E$ occurs after γ_M in the prescribed enumeration.

Proof. For each $\gamma \in E$ and each $i \in \{1, 2, \dots, M\}$, by Lemma 2.1, $A^J \gamma = \gamma_i$ holds for at most one $J \in \mathbb{N}_0$. Thus, there are at most $M|E|$ integers J for which $A^J E \cap \{\gamma_1, \gamma_2, \dots, \gamma_M\} \neq \emptyset$. Since the $M|E| + 1$ integers $0, 1, \dots, M|E|$ therefore cannot all yield a nonempty intersection, at least one of them satisfies (5.6). Finally, $A^J \gamma \neq 1$ for $\gamma \neq 1$ by injectivity of A , so (5.6) says exactly that every character of $A^J E$ occurs after γ_M in the enumeration. $\square$

We shall use the index set $\mathcal{I} := \{(a, b) \in \mathbb{N}^2 : 1 \leq b \leq K_a\}$. Now we define $J_{n,s}$ recursively. Assume $J_{a,b}$ has been determined for all $(a, b) \in \mathcal{I}$ with $(a, b) < (n, s)$ in the lexicographical order. We record the Fourier positions that have already been occupied by the previous blocks:

$$(5.7) \quad \mathcal{O}_{n,s} := \bigcup_{(a,b) \in \mathcal{I} : (a,b) < (n,s)} A^{J_{a,b}} E_a.$$

Since $\mathcal{O}_{n,s}$ is finite, we define $M_{n,s} := \max\{r \in \mathbb{N} : \gamma_r \in \mathcal{O}_{n,s}\}$, with the convention that $M_{n,s} := 0$ if $\mathcal{O}_{n,s} = \emptyset$. By Lemma 5.2, at least one integer $J \in \{0, 1, \dots, M_{n,s}|E_n|\}$ satisfies

$$(5.8) \quad A^J E_n \cap \{\gamma_1, \gamma_2, \dots, \gamma_{M_{n,s}}\} = \emptyset;$$

we let $J_{n,s}$ be the least such J . Thus all characters in $A^{J_{n,s}} E_n$ are placed strictly after all previously occupied frequency positions.

By the recursive choice of $J_{n,s}$, the family of *Fourier blocks* $A^{J_{n,s}} E_n$, indexed by the pairs (n, s) with $n \in \mathbb{N}$ and $1 \leq s \leq K_n$, has the following two properties:

- (B1) the blocks are pairwise disjoint;
- (B2) if $(a, b) \in \mathcal{I}$ with $(a, b) < (n, s)$ in the lexicographic order, then every character in $A^{J_{n,s}} E_n$ occurs strictly later than every character in $A^{J_{a,b}} E_a$ with respect to the enumeration (2.3).

The proofs of Propositions 5.3 and 5.4 below use the integers $J_{n,s}$ only through (B1) and (B2). Accordingly, in Subsection 5.2 the recursive choice above is replaced, in each of three concrete settings, by an explicit prescription, and all that has to be checked is that (B1) and (B2) still hold.

For $N \in \mathbb{N}$, define $\tilde{F}_N := \sum_{n=1}^N \tilde{\Delta}_n$. Since $\|\tilde{\Delta}_N\|_\infty \leq \|\Delta_N\|_\infty = \alpha_N$, it follows from (5.2) that $\{\tilde{F}_N\}_{N \in \mathbb{N}}$ converges uniformly; we denote its limit by $\tilde{F}$. Each $\tilde{F}_N$ is a finite linear combination of characters, hence continuous, so $\tilde{F} \in C(G, \mathbb{R})$.

Proposition 5.3 (Fourier series in a prescribed order). *There exists a sequence $\{c_r\}_{r \in \mathbb{N}}$ in $\mathbb{Q}$ such that $c_r = \widehat{\tilde{F}}(\gamma_r)$ for all $r \in \mathbb{N}$ and that*

$$(5.9) \quad \tilde{F}(x) = \sum_{r=1}^{+\infty} c_r \gamma_r(x),$$

with the partial sums converging uniformly on G in the enumeration order (2.3).

Proof. Recall from (B1) that for $n \in \mathbb{N}$ and $1 \leq s \leq K_n$, the Fourier blocks $A^{J_{n,s}} E_n$ are pairwise disjoint. If $\gamma_r = A^{J_{n,s}} \gamma$ for some $n \in \mathbb{N}$, $1 \leq s \leq K_n$, and $\gamma \in E_n$, then set

$$(5.10) \quad c_r := \frac{1}{K_n} \widehat{\Delta}_n(\gamma);$$

otherwise set $c_r := 0$. This is well-defined: by (B1) at most one pair (n, s) can occur, and $A^{J_{n,s}}$ is injective by Lemma 2.1, so γ is then determined by γ_r .

We now prove the uniform convergence of the Fourier partial sums. For each subset $E \subseteq E_n$,

$$\left\| \frac{1}{K_n} \sum_{\gamma \in E} \widehat{\Delta}_n(\gamma) A^{J_{n,s}} \gamma \right\|_\infty \leq \frac{1}{K_n} \sum_{\gamma \in E} |\widehat{\Delta}_n(\gamma)| \leq \frac{\|\Delta_n\|_A}{K_n} < 2^{-n}.$$

Hence, a cutoff inside a single Fourier block creates an error of at most 2^{-n} .

For $R \in \mathbb{N}$, if the cutoff at γ_R falls inside the block corresponding to $\tilde{\Delta}_{n,s_0}$, then the remaining part is bounded as follows:

$$(5.11) \quad \left\| \tilde{F} - \sum_{r=1}^R c_r \gamma_r \right\|_{\infty} \leq 2^{-n} + \sum_{s=s_0+1}^{K_n} \|\tilde{\Delta}_{n,s}\|_{\infty} + \sum_{j=n+1}^{+\infty} \|\tilde{\Delta}_j\|_{\infty} \leq 2^{-n} + \alpha_n + \sum_{j=n+1}^{+\infty} \alpha_j.$$

If the cutoff lies between two consecutive blocks, the first term is absent.

Write $g(n) := 2^{-n} + \sum_{j=n}^{+\infty} \alpha_j$, so that (5.11) gives $\|\tilde{F} - \sum_{r=1}^R c_r \gamma_r\|_{\infty} \leq g(n)$ whenever the cutoff at γ_R lies in, or immediately after, a block with first index $n \in \mathbb{N}$. By (5.2), g is nonincreasing and $g(n) \rightarrow 0$ as $n \rightarrow +\infty$. For each n the set $\bigcup_{(a,b) \leq (n, K_n)} A^{J_{a,b}} E_a$ is finite, so for all sufficiently large $R \in \mathbb{N}$ the cutoff lies beyond every block with first index at most n ; hence $\|\tilde{F} - \sum_{r=1}^R c_r \gamma_r\|_{\infty} \leq g(n)$ for all such R , and $\sum_{r=1}^R c_r \gamma_r \rightarrow \tilde{F}$ uniformly on G as $R \rightarrow +\infty$.

Since this convergence is uniform and the characters are orthonormal in $L^2(m_G)$, term-by-term integration gives $c_r = \widehat{\tilde{F}}(\gamma_r)$ for all $r \in \mathbb{N}$, and $\widehat{\tilde{F}}(1) = 0$; thus (5.9) is the Fourier series of $\tilde{F}$, listed in the prescribed order, as claimed. $\square$

Since relocation does not change integrals against invariant measures, $\tilde{F}$ inherits the exposure property of F :

Proposition 5.4 (A Fourier series exposure). *The normalised Haar measure m_G is the unique T -invariant maximizing measure for $\tilde{F}$ defined above.*

Proof. By (5.4), (5.1), and $\tilde{F}_N = \sum_{n=1}^N \tilde{\Delta}_n$, we have $\int \tilde{F}_N d\nu = \int F_N d\nu$ for all $\nu \in \mathcal{M}(G, T)$. Letting $N \rightarrow +\infty$ and using the uniform convergence of $\{F_N\}_{N \in \mathbb{N}}$ and $\{\tilde{F}_N\}_{N \in \mathbb{N}}$ together with $\nu(G) = 1$, we obtain $\int \tilde{F} d\nu = \int F d\nu$. The conclusion now follows from Theorem D. $\square$

Finally, the coefficients c_r can be computed exactly, provided that the enumeration and the integers $J_{n,s}$ are effective:

Proposition 5.5 (Exact computability of the coefficients). *Suppose the enumeration (2.3) is chosen so that, given $r \in \mathbb{N}$, the character γ_r and the integers $J_{n,s}$ can be produced by finite procedures, and that equality in Γ is decidable. Then there is an algorithm which, on input $r \in \mathbb{N}$, returns the exact value of $c_r \in \mathbb{Q}$ after finitely many operations in the sense of Remark 1.2, generating at most r Fourier blocks.*

Proof. Every quantity in the recursion of Subsection 4.2 is a rational number or a positive integer obtained from earlier data by finitely many operations of types (O1)–(O5): this is the content of (4.3), (4.4), (2.5), (4.10), (4.11), and (4.14), together with Lemmas 2.4 and 3.3. In particular, for each $n \in \mathbb{N}$ the complete finite Fourier expansion of F_n , hence of $\Delta_n = F_n - F_{n-1}$, is produced exactly, and so are $\|\Delta_n\|_A \in \mathbb{Q}$ and $K_n = 1 + \lceil 2^n \|\Delta_n\|_A \rceil$. Enumerate $\mathcal{I}$ in lexicographic order as $(n_1, s_1), (n_2, s_2), \dots$, which is possible since each K_a is finite, and write $\mathcal{E}_i := A^{J_{n_i, s_i}} E_{n_i}$. By Lemma 5.1 each E_n is nonempty, and A is injective, so every $\mathcal{E}_i$ is nonempty; set $\mu(i) := \min\{r \in \mathbb{N} : \gamma_r \in \mathcal{E}_i\}$. By (B2) the integers $\mu(1), \mu(2), \dots$ are strictly increasing, so $\mu(i) \geq i$ for all $i \in \mathbb{N}$, and therefore

$$\mathcal{E}_i \cap \{\gamma_1, \gamma_2, \dots, \gamma_r\} = \emptyset \quad \text{for all } i > r.$$

On input r it is thus enough to generate $\mathcal{E}_1, \dots, \mathcal{E}_r$, that is, at most r Fourier blocks, for which only the first $n_r \leq r$ steps of the recursion of Subsection 4.2 are needed. Testing whether γ_r lies in one of these finitely many finite sets uses only decidability of equality in Γ . By (B1) at most one i occurs, and (5.10) gives $c_r = K_{n_i}^{-1} \widehat{\Delta_{n_i}}(\gamma)$ if $\gamma_r = A^{J_{n_i, s_i}} \gamma$ with $\gamma \in E_{n_i}$ and $i \leq r$, and $c_r = 0$ otherwise. $\square$

5.2. Effective Fourier series construction. The integer $J_{n,s}$ in (5.8) is found by a finite search. Hence Propositions 5.3, 5.4, and 5.5 apply to every enumeration (2.3) for which γ_r can be produced from r by a finite procedure and equality in Γ is decidable, and this is all that is needed for Theorem B. For expanding circle endomorphisms and the one-sided full shift, where A multiplies each frequency index by a fixed integer, the integers $J_{n,s}$ can moreover be prescribed by explicit recursive formulas. In these two settings we follow the same three steps: we introduce a gauge on $\mathcal{F}_{\mathbb{Q}}$ (the *degree*, or the *depth*) which is controlled under the four closure properties of Lemma 2.4; we use it to bound the frequency support of Δ_n and hence of its relocated copies; and we then choose the integer $J_{n,s}$ so that consecutive blocks are strictly separated, which gives (B1) and (B2) and allows Propositions 5.3 and 5.4 to apply.

5.2.1. Expanding endomorphisms on the circle: proof of Theorem A. Here we prove Theorem A. We choose the enumeration in (2.3) by setting

$$(5.12) \quad \gamma_{2k-1}(x) := \exp(2\pi i k x), \quad \gamma_{2k}(x) := \exp(-2\pi i k x),$$

for $k \in \mathbb{N}$. On $\mathbb{T}$, inversion-invariance of the Fourier coefficients is the classical condition for a function to be even, so every $P \in \mathcal{F}_{\mathbb{Q}}$ can be written as

$$(5.13) \quad P(x) = a_0 + \sum_{k=1}^N a_k \cos(2\pi k x),$$

where $a_0, a_1, \dots, a_N \in \mathbb{Q}$ and $N \in \mathbb{N}_0$.

We define the *nonnegative Fourier support* of P to be $\{k \in \mathbb{N}_0 : a_k \neq 0\}$, and set $\deg P := \max(\{k \in \mathbb{N} : a_k \neq 0\} \cup \{0\})$. We require the following simple lemma, whose proof is left to the reader.

Lemma 5.6. *For $P, Q \in \mathcal{F}_{\mathbb{Q}}$, $R \in \mathbb{Q}[t] \setminus \{0\}$, $j \in \mathbb{N}_0$, and the expanding map $T(x) = dx \pmod{1}$ on $\mathbb{T}$, with $d \in \mathbb{Z}$, $|d| \geq 2$, the following hold:*

- (i) $\deg(P + Q) \leq \max\{\deg P, \deg Q\}$;
- (ii) $\deg(P \cdot Q) \leq \deg P + \deg Q$;
- (iii) $\deg(P \circ T^j) = |d|^j \deg P$;
- (iv) $\deg(R \circ P) = \deg R \cdot \deg P$.

Proof of Theorem A. We prescribe a sequence $\{e_n\}_{n \in \mathbb{N}_0}$ recursively so that $\deg F_n \leq e_n$ for all $n \in \mathbb{N}_0$. Set $e_0 := 1$. By Subsection 4.2, $\deg F_0 = 0 \leq e_0$. Now assume that $\deg F_{n-1} \leq e_{n-1}$ for some $n \in \mathbb{N}$. By Definition 2.5, for each $k \in \mathbb{N}$ and each $r \in \{2k-1, 2k\}$, the nonzero indices in the cosine expansion of $K_{r,N}$ in (5.13) are of the form $|d^j - d^\ell|k$ with integers $0 \leq \ell < j < N$. For such k and r , we have $k \leq r$, while $|d^j - d^\ell| \leq 2|d|^{N-1} \leq |d|^N$ since $|d| \geq 2$; hence $\deg K_{r,N} \leq r|d|^N$ and so $\deg \mathcal{D}_n \leq (n+1)|d|^{2^{n+2}}$ by Subsection 4.1, (2.6), (2.5), and Lemma 5.6. It follows from (4.7), Lemma 5.6 (i), and the induction hypothesis that

$$(5.14) \quad \deg H_n \leq \max\{e_{n-1}, (n+1)|d|^{2^{n+2}}\}.$$

Using (3.1), (4.11), and Lemma 5.6, we obtain

$$(5.15) \quad \deg(\mathcal{B}_{d_n, \overline{B}_n}(S_{M_n} H_n / M_n)) \leq e_n,$$

where we define

$$e_n := d_n |d|^{M_n} \max\{e_{n-1}, (n+1)|d|^{2^{n+2}}\}.$$

By (4.12), (5.14), (5.15), and Lemma 5.6 (i), $\deg F_n \leq e_n$, which completes the induction.

For each $n \in \mathbb{N}$, the definition of Δ_n and Lemma 5.6 (i) give $\deg \Delta_n \leq \max\{\deg F_n, \deg F_{n-1}\} \leq \max\{e_n, e_{n-1}\} = e_n$. Moreover, Δ_n has zero Haar mean by Proposition 4.1. Thus the nonnegative Fourier support of Δ_n is contained in $\{1, 2, \dots, e_n\}$. Since composition with T^J multiplies every

positive Fourier frequency by $|d|^J$, the nonnegative Fourier support of $\Delta_n \circ T^J$ is contained in $[|d|^J, |d|^J e_n]$. We choose

$$J_{1,1} := 0, \quad J_{n,s+1} := J_{n,s} + 1 + \lceil \log_{|d|}(e_n) \rceil, \quad \text{and } J_{n,K_n+1} := J_{n+1,1},$$

where $n, s \in \mathbb{N}$ and $1 \leq s \leq K_n$. The integers $J_{n,s}$ are defined recursively by the formulas above, so Proposition 5.5 applies. With the above choice of the $J_{n,s}$, we have $|d|^{J_{n,s+1}} > e_n |d|^{J_{n,s}}$: the relocated increment $\tilde{\Delta}_{n,s+1}$ defined in (5.4) (respectively $\tilde{\Delta}_{n+1,1}$, when $s = K_n$) has nonnegative Fourier support contained in $[|d|^{J_{n,s+1}}, +\infty)$, strictly beyond the nonnegative Fourier support of every relocated increment already placed, which is contained in $[0, e_n |d|^{J_{n,s}}]$. By (5.4) and (5.5), $A^{J_{n,s}} E_n$ is precisely the frequency support of $\tilde{\Delta}_{n,s}$. By (5.12), the pairs of characters corresponding to $\{k, -k\}$ are ordered by increasing k ; it follows that the blocks $A^{J_{n,s}} E_n$, $n \in \mathbb{N}$, $1 \leq s \leq K_n$, are pairwise disjoint, and that consecutive blocks $A^{J_{n,s}} E_n$ in the lexicographic order of (n, s) occur strictly successively with respect to (2.3). Thus (B1) and (B2) hold, so Proposition 5.3 applies: $\tilde{F} = \sum_{r=1}^{+\infty} \beta_r \gamma_r$ with $\beta_r := \widehat{\tilde{F}}(\gamma_r) \in \mathbb{Q}$, the partial sums converging uniformly in the enumeration order.

Each $\tilde{\Delta}_n$ lies in $\mathcal{F}_{\mathbb{Q}}$ by Lemma 2.4, so $\widehat{\tilde{F}}(-k) = \widehat{\tilde{F}}(k) \in \mathbb{Q}$ for all $k \in \mathbb{N}$. By our choice of enumeration, for all $k \in \mathbb{N}$, $\beta_{2k-1} = \widehat{\tilde{F}}(k)$ and $\beta_{2k} = \widehat{\tilde{F}}(-k)$. Set $c_k := 2\widehat{\tilde{F}}(k) \in \mathbb{Q}$. Then $c_k \cos(2\pi kx) = \beta_{2k-1} \gamma_{2k-1}(x) + \beta_{2k} \gamma_{2k}(x)$. Hence for all $K \in \mathbb{N}$, the partial sum $\sum_{r=1}^{2K} \beta_r \gamma_r$ of Proposition 5.3 is equal to $\sum_{k=1}^K c_k \cos(2\pi kx)$; being a subsequence of a uniformly convergent sequence, these cosine partial sums converge uniformly to $\tilde{F}$. Together with Proposition 5.4, this proves Theorem A. $\square$

5.2.2. The one-sided full shift: proof of Theorem C. Here we prove Theorem C. We take the enumeration in (2.3) to be the Walsh ordering of (1.3), namely $\gamma_r := w_r$ for $r \in \mathbb{N}$. Since each w_r is real-valued with $w_r^{-1} = w_r$, every $P \in \mathcal{F}_{\mathbb{Q}}$ has the form $P = \sum_{k=0}^N a_k w_k$ with $a_0, a_1, \dots, a_N \in \mathbb{Q}$. It will be convenient to define

$$\text{depth } P := \min\{p \in \mathbb{N}_0 : P(x) = P(y) \text{ whenever } x, y \in X \text{ and } x_j = y_j \text{ for } 0 \leq j < p\}.$$

We require the following simple lemma, whose proof is left to the reader.

Lemma 5.7. *For $P, Q \in \mathcal{F}_{\mathbb{Q}}$, $R \in \mathbb{Q}[t] \setminus \{0\}$, $j \in \mathbb{N}_0$, and the one-sided shift T on $X := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}_0}$, the following hold:*

- (i) $\text{depth}(P + Q) \leq \max\{\text{depth } P, \text{depth } Q\}$;
- (ii) $\text{depth}(P \cdot Q) \leq \max\{\text{depth } P, \text{depth } Q\}$;
- (iii) $\text{depth}(P \circ T^j) \leq j + \text{depth } P$;
- (iv) $\text{depth}(R \circ P) \leq \text{depth } P$.

Proof of Theorem C. We prescribe a sequence $\{e_n\}_{n \in \mathbb{N}_0}$ recursively so that $\text{depth } F_n \leq e_n$ for all $n \in \mathbb{N}_0$. Set $e_0 := 1$. Then by Subsection 4.2, $\text{depth } F_0 = 0 \leq e_0$. Now assume that $\text{depth } F_{n-1} \leq e_{n-1}$ for some $n \in \mathbb{N}$. By the definition of the Walsh ordering, $\text{depth } w_r = 1 + \lfloor \log_2 r \rfloor$ for $r \in \mathbb{N}$. It follows from Definition 2.5 and Lemma 5.7 that for all $r, N \in \mathbb{N}$, $\text{depth } K_{r,N} \leq N + \lfloor \log_2 r \rfloor$, and so $\text{depth } \mathcal{D}_n \leq 2^{n+2} + \lfloor \log_2(n+1) \rfloor$ by Subsection 4.1, (2.6), (2.5), and Lemma 5.7. It follows from (4.7), Lemma 5.7 (i), and the induction hypothesis that

$$(5.16) \quad \text{depth } H_n \leq \max\{e_{n-1}, 2^{n+2} + \lfloor \log_2(n+1) \rfloor\}.$$

Using (3.1), (4.11), and Lemma 5.7, we obtain

$$(5.17) \quad \text{depth}(\mathcal{B}_{d_n, \bar{B}_n}(S_{M_n} H_n / M_n)) \leq e_n,$$

where we define $e_n := \max\{e_{n-1}, 2^{n+2} + \lfloor \log_2(n+1) \rfloor\} + M_n - 1$. By (4.12), (5.16), (5.17), and Lemma 5.7 (i), $\text{depth } F_n \leq e_n$, which completes the induction.

By the definition of Δ_n and Lemma 5.7 (i), $\text{depth } \Delta_n \leq \max\{\text{depth } F_n, \text{depth } F_{n-1}\} \leq e_n$ for each $n \in \mathbb{N}$. Moreover, a Walsh polynomial of depth at most e_n involves only w_k with $0 \leq k \leq 2^{e_n} - 1$. Since Δ_n has zero Haar mean for each $n \in \mathbb{N}$ by Proposition 4.1, its frequency support is contained in $\{w_k : k \in \mathbb{N}, 1 \leq k \leq 2^{e_n} - 1\}$. It follows directly from the definition that $w_k \circ T^J = w_{2^J k}$ for all $k, J \in \mathbb{N}_0$. Thus for $n \in \mathbb{N}$, the frequency support of $\Delta_n \circ T^J$ is contained in $\{w_k : 2^J \leq k \leq (2^{e_n} - 1)2^J\}$. We choose

$$J_{1,1} := 0, \quad J_{n,s+1} := J_{n,s} + e_n, \quad \text{and } J_{n,K_n+1} := J_{n+1,1},$$

where $n, s \in \mathbb{N}$ and $1 \leq s \leq K_n$. The integers $J_{n,s}$ are defined recursively by the formulas above, so Proposition 5.5 applies. With this choice, $2^{J_{n,s+1}} > (2^{e_n} - 1)2^{J_{n,s}}$ for each $n \in \mathbb{N}$ and each integer $1 \leq s \leq K_n$: the block $A^{J_{n,s+1}}E_n$ (respectively $A^{J_{n+1,1}}E_{n+1}$, when $s = K_n$) has Walsh-frequency support in $\{w_k : k \in \mathbb{N}, k \geq 2^{J_{n,s+1}}\}$, strictly beyond the Walsh-frequency support in $\{w_k : k \in \mathbb{N}, 1 \leq k \leq (2^{e_n} - 1)2^{J_{n,s}}\}$ of every block already placed. Since $\gamma_r = w_r$ for all $r \in \mathbb{N}$, it follows that the blocks $A^{J_{n,s}}E_n$, $n \in \mathbb{N}$, $1 \leq s \leq K_n$, are pairwise disjoint, and that consecutive blocks $A^{J_{n,s}}E_n$ in the lexicographic order of (n, s) occur strictly successively with respect to (2.3). Thus (B1) and (B2) hold, and Theorem C follows from Propositions 5.3 and 5.4. $\square$

5.2.3. Ergodic toral endomorphisms: proof of Theorem B. Here we prove Theorem B. The dual group of $\mathbb{T}^m$ is identified with $\mathbb{Z}^m$, the frequency $k \in \mathbb{Z}^m$ corresponding to the character $x \mapsto \exp(2\pi i \langle k, x \rangle)$, and composition with T_B maps the character with frequency k to the character with frequency $B^t k$. We choose the enumeration in (2.3) by ordering $\mathbb{Z}^m \setminus \{0\}$ first by increasing $\|k\|_\infty$ and then lexicographically, and write k_r for the frequency of γ_r . Unlike in the two settings above, $\|(B^t)^J k\|_\infty$ need not increase with J if B^t has a contracting direction, so we keep the integers $J_{n,s}$ chosen in (5.8).

Proof of Theorem B. For $R \in \mathbb{N}_0$ set $L(R) := (2R + 1)^m - 1$, the number of $k \in \mathbb{Z}^m \setminus \{0\}$ with $\|k\|_\infty \leq R$. Since $L(r) \geq r$, the enumeration satisfies $\|k_r\|_\infty \leq r$, so k_r is found from r by listing the finitely many $k \in \mathbb{Z}^m$ with $\|k\|_\infty \leq r$ in the prescribed order. Equality of characters is equality in $\mathbb{Z}^m$, and A acts on frequencies by the integer matrix B^t . Consequently the integers $J_{n,s}$ of (5.8) are produced by finite procedures, and Propositions 5.3, 5.4, and 5.5 apply. Since the enumeration is ordered by increasing ℓ^∞ -norm, for each $R \in \mathbb{N}_0$ the square partial sum $\Sigma_R \tilde{F}$ of (1.2) coincides with $\sum_{r=1}^{L(R)} c_r \gamma_r$, where the $c_r \in \mathbb{Q}$ are given by Proposition 5.3. The square partial sums therefore form a subsequence of the partial sums in the prescribed enumeration, and converge uniformly to $\tilde{F}$. Together with Propositions 5.4 and 5.5, this proves Theorem B, the coefficient of $\exp(2\pi i \langle k_r, x \rangle)$ being c_r . $\square$

6. RIESZ PRODUCTS AND THE WIENER ALGEBRA OBSTRUCTION

In this section we prove Theorem E. For $f \in C(G, \mathbb{R}) \cap A(G)$, the integral $\int f d\nu$ depends on $\nu \in \mathcal{M}(G, T)$ only through the Fourier coefficients of ν at one representative character from each A -orbit in Γ , weighted by the corresponding orbit sums of $\hat{f}$, defined in (6.1); this is (6.2) below. If the orbit sums over all nontrivial A -orbits vanish, then $\int f d\nu = \hat{f}(1)$ for all $\nu \in \mathcal{M}(G, T)$, so that $f - \hat{f}(1)$ is a weak coboundary. It therefore suffices to show that if the orbit sum over some nontrivial A -orbit is nonzero, then m_G is not f -maximizing; this is Proposition 6.5. Its proof constructs an invariant measure whose Fourier coefficient at that orbit has a prescribed phase and dominates all its other Fourier coefficients. The tool for this is the Riesz product, which originated in the work of Riesz [41] on the circle and was extended to compact abelian groups by Hewitt and Zuckerman [27]: built on a dissociate sequence of characters, here supplied by Lemma 6.4, it is a measure whose Fourier coefficients are given explicitly. We follow the formulation in [16, Section 4].

Definition 6.1 (Dissociate sequences). An infinite sequence of distinct characters $\Lambda = \{\gamma_n\}_{n \in \mathbb{N}_0}$ in Γ is said to be *dissociate* if for each $k \in \mathbb{N}$ and all k -tuples $(n_1, n_2, \dots, n_k) \in \mathbb{N}_0^k$ of distinct nonnegative integers, the equality

$$\gamma_{n_1}^{\varepsilon_1} \gamma_{n_2}^{\varepsilon_2} \cdots \gamma_{n_k}^{\varepsilon_k} = 1,$$

with $\varepsilon_j \in \{-2, -1, 0, 1, 2\}$ for all $1 \leq j \leq k$, implies that $\gamma_{n_1}^{\varepsilon_1} = \gamma_{n_2}^{\varepsilon_2} = \cdots = \gamma_{n_k}^{\varepsilon_k} = 1$. Equivalently, Λ is dissociate if every character in Γ can be represented in at most one way as a finite product $\gamma_{n_1}^{\tau_1} \gamma_{n_2}^{\tau_2} \cdots \gamma_{n_k}^{\tau_k}$ of elements of Λ , where the integers n_j are distinct and $\tau_j \in \{-1, 0, 1\}$, except that the exponents 1 and -1 are identified whenever $\gamma_{n_j}^2 = 1$. Representations are understood up to reordering the factors and omission of identity factors. The empty product represents 1.

In the following, the real part of a complex number z is denoted by $\Re(z)$. We recall the standard Riesz product construction; see [27, Theorem 3.2].

Lemma 6.2 (Riesz product). *Let $\{\gamma_n\}_{n \in \mathbb{N}_0}$ be a dissociate sequence of nontrivial characters in Γ . Let $\{a_n\}_{n \in \mathbb{N}_0}$ be a sequence in $\mathbb{C}$ satisfying $|a_n| \leq 1$ for all $n \in \mathbb{N}_0$. For $N \in \mathbb{N}_0$, let $\mu^{(N)}$ be the measure on G that is absolutely continuous with respect to Haar measure m_G with density*

$$\frac{d\mu^{(N)}}{dm_G}(x) = \prod_{n=0}^N (1 + \Re(a_n \gamma_n(x))).$$

Then $\{\mu^{(N)}\}_{N \in \mathbb{N}_0}$ is a sequence of probability measures converging in the weak topology to some probability measure μ as $N \rightarrow +\infty$.*

Remark 6.3. In the setting of Lemma 6.2, the limiting measure μ is called the Riesz product, and, for $a = \{a_n\}_{n \in \mathbb{N}_0}$, we write formally

$$\mu_a := \prod_{n=0}^{+\infty} (1 + \Re(a_n \gamma_n(x))).$$

For the construction in Proposition 6.5, we need, for each nontrivial $\gamma \in \Gamma$, a set $S \subseteq \mathbb{N}_0$ of positive lower density such that $\{A^m \gamma\}_{m \in S}$ is dissociate. In fact, S may be chosen to be an arithmetic progression $L\mathbb{N}_0$ for some $L \in \mathbb{N}$.

Lemma 6.4 (Dissociate subsequence). *Let $\gamma \in \Gamma$ be a nontrivial character. Then there exists $L \in \mathbb{N}$ such that the sequence $\{A^{Ln} \gamma\}_{n \in \mathbb{N}_0}$ is dissociate.*

Proof. We deduce the lemma from a theorem of Lind [35]. We first recall the relevant terminology, in multiplicative notation and in the form needed here. Let H be an abelian group with identity 1. A sequence $\{E_n\}_{n \in \mathbb{N}_0}$ of subsets of H is *independent* if the equality $e_1 e_2 \cdots e_k = 1$, where $k \in \mathbb{N}$ and $e_j \in E_{n_j}$ for pairwise distinct $n_1, n_2, \dots, n_k \in \mathbb{N}_0$, implies that $e_1 = e_2 = \cdots = e_k = 1$. An automorphism θ of H is *aperiodic* if for $p \in \mathbb{N}$ and $h \in H$, the equality $\theta^p h = h$ implies that $h = 1$. An aperiodic automorphism θ of H is *lacunarily independent* if for each finite set $E \subseteq H$, there exists $L \in \mathbb{N}$ such that the sequence $\{\theta^{m_n} E\}_{n \in \mathbb{N}_0}$ is independent whenever $\{m_n\}_{n \in \mathbb{N}_0}$ is a sequence of integers with $m_{n+1} - m_n \geq L$ for all $n \in \mathbb{N}_0$. Lind proved that every aperiodic automorphism of a countable abelian group is lacunarily independent; see [35, Theorem 2].

By Lemma 2.1, A is injective and 1 is its only periodic point. Since A need not be surjective, we first embed Γ in a larger group to which A extends as an automorphism. For $(\chi, n), (\chi', n') \in \Gamma \times \mathbb{N}_0$, write $(\chi, n) \sim (\chi', n')$ if $A^{n'} \chi = A^n \chi'$. This is an equivalence relation, transitivity following from the injectivity of A . Let $\tilde{\Gamma} := (\Gamma \times \mathbb{N}_0) / \sim$ denote the set of equivalence classes, and write $[\chi, n]$ for the class of $(\chi, n) \in \Gamma \times \mathbb{N}_0$, which should be thought of as $A^{-n} \chi$. One checks directly that

$$[\chi, n] \cdot [\chi', n'] := [A^{n'} \chi \cdot A^n \chi', n + n']$$

is a well-defined operation making $\tilde{\Gamma}$ an abelian group, with identity $[1, 0]$ and $[\chi, n]^{-1} = [\chi^{-1}, n]$. Similarly, $\tilde{A}[\chi, n] := [A\chi, n]$ defines an automorphism $\tilde{A}$ of $\tilde{\Gamma}$, with inverse $[\chi, n] \mapsto [\chi, n+1]$. Define $\iota: \Gamma \rightarrow \tilde{\Gamma}$ by $\iota(\chi) := [\chi, 0]$. Then ι is a homomorphism such that $\tilde{A} \circ \iota = \iota \circ A$. Moreover, ι is injective, as $(\chi, 0) \sim (\chi', 0)$ only if $\chi = \chi'$ by definition of $\sim$. In other words, $\tilde{\Gamma}$ is the direct limit of $\Gamma \xrightarrow{A} \Gamma \xrightarrow{A} \dots$. Dually, passing from A to $\tilde{A}$ corresponds to passing from T to its natural extension. Since Γ is countable (see the discussion before (2.3)), the quotient $\tilde{\Gamma} = (\Gamma \times \mathbb{N}_0)/\sim$ is at most countable, and is in fact countable since the embedding $\iota: \Gamma \rightarrow \tilde{\Gamma}$ exhibits the infinite subset $\iota(\Gamma)$ of $\tilde{\Gamma}$. Moreover, $\tilde{A}$ is aperiodic: if $\tilde{A}^p[\chi, n] = [\chi, n]$ for some $p \in \mathbb{N}$ and $[\chi, n] \in \tilde{\Gamma}$, then $A^{n+p}\chi = A^n\chi$, so $\chi = 1$ by Lemma 2.1, and hence $[\chi, n] = [1, 0]$.

Now set $E := \{\iota(\gamma^\varepsilon) : \varepsilon \in \{-2, -1, 0, 1, 2\}\}$. By Lind's theorem [35, Theorem 2], $\tilde{A}$ is lacunarily independent, so there exists $L \in \mathbb{N}$ such that $\{\tilde{A}^{Ln}E\}_{n \in \mathbb{N}_0}$ is independent. Let $k \in \mathbb{N}$, let $n_1, n_2, \dots, n_k \in \mathbb{N}_0$ be pairwise distinct, and let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{-2, -1, 0, 1, 2\}$ satisfy $\prod_{j=1}^k (A^{Ln_j}\gamma)^{\varepsilon_j} = 1$. For each integer $1 \leq j \leq k$, $e_j := \iota((A^{Ln_j}\gamma)^{\varepsilon_j})$ is equal to $\tilde{A}^{Ln_j}\iota(\gamma^{\varepsilon_j}) \in \tilde{A}^{Ln_j}E$, because A and ι are homomorphisms with $\iota \circ A = \tilde{A} \circ \iota$. Moreover, $e_1 e_2 \dots e_k = \iota(1) = [1, 0]$. By independence, each e_j equals $[1, 0] = \iota(1)$, and the injectivity of ι gives $(A^{Ln_j}\gamma)^{\varepsilon_j} = 1$. As $\gamma \neq 1$, the characters $A^{Ln}\gamma$, $n \in \mathbb{N}_0$, are nontrivial and pairwise distinct by Lemma 2.1, and so $\{A^{Ln}\gamma\}_{n \in \mathbb{N}_0}$ is dissociate. $\square$

We group the characters in Γ according to the orbit under iterations of A . Namely, write $\gamma \sim \gamma'$ if $A^m\gamma = A^n\gamma'$ for some $m, n \in \mathbb{N}_0$. By injectivity of A , this is an equivalence relation on Γ . Let $\mathcal{O} := \Gamma/\sim$ denote the set of orbits, and fix a representative $\gamma_{\mathcal{O}} \in \mathcal{O}$ for each $\mathcal{O} \in \mathcal{O}$. We write $\mathcal{O}_0 := \{1\}$ for the trivial orbit. The equivalence relation is compatible with inversion: if $\gamma \sim \gamma'$, then $\gamma^{-1} \sim (\gamma')^{-1}$. Hence for $\mathcal{O} \in \mathcal{O}$, we write $\mathcal{O}^{-1} := \{\gamma^{-1} : \gamma \in \mathcal{O}\}$, which is again an orbit. For each pair $\mathcal{O}, \mathcal{O}^{-1}$, with $\mathcal{O} \neq \mathcal{O}^{-1}$, we choose the representatives so that $\gamma_{\mathcal{O}^{-1}} = \gamma_{\mathcal{O}}^{-1}$.

For $f \in A(G)$ and each $\mathcal{O} \in \mathcal{O}$, define the *orbit sum*

$$(6.1) \quad b_{\mathcal{O}}(f) := \sum_{\gamma \in \mathcal{O}} \hat{f}(\gamma).$$

Since $|b_{\mathcal{O}}(f)| \leq \sum_{\gamma \in \mathcal{O}} |\hat{f}(\gamma)|$, we have $\sum_{\mathcal{O} \in \mathcal{O}} |b_{\mathcal{O}}(f)| < +\infty$. Hence for all $\nu \in \mathcal{M}(G, T)$, absolute convergence and (2.1) give

$$(6.2) \quad \int f \, d\nu = \sum_{\gamma \in \Gamma} \hat{f}(\gamma) \int \gamma \, d\nu = \hat{f}(1) + \sum_{\mathcal{O} \in \mathcal{O} \setminus \{\mathcal{O}_0\}} b_{\mathcal{O}}(f) \hat{\nu}(\gamma_{\mathcal{O}}^{-1}).$$

The following result implies that if m_G is f -maximizing, then $b_{\mathcal{O}}(f) = 0$ for all nontrivial orbits.

Proposition 6.5 (A nonvanishing orbit sum prevents Haar measure from being maximizing). *Let $f \in C(G, \mathbb{R}) \cap A(G)$. Suppose $\mathcal{O} \in \mathcal{O}$ is a nontrivial orbit with $b_{\mathcal{O}}(f) \neq 0$. Then there exists $\nu \in \mathcal{M}(G, T)$ such that*

$$\int f \, dm_G < \int f \, d\nu.$$

Proof. We distinguish two cases according to whether $\mathcal{O} = \mathcal{O}^{-1}$ or not.

Case 1. $\mathcal{O} \neq \mathcal{O}^{-1}$. By Lemma 6.4, choose $L \in \mathbb{N}$ such that $\{A^{LN}\gamma_{\mathcal{O}}\}_{N \in \mathbb{N}_0}$ is dissociate, and set $\sigma := b_{\mathcal{O}}(f)/|b_{\mathcal{O}}(f)|$. For $t \in (0, 1]$, let $\mu_{t, \mathcal{O}}$ be the Riesz product given by Lemma 6.2,

$$\mu_{t, \mathcal{O}} := \prod_{N=0}^{+\infty} (1 + \Re(t\sigma A^{LN}\gamma_{\mathcal{O}})) = \prod_{N=0}^{+\infty} \left(1 + \frac{t\sigma}{2} A^{LN}\gamma_{\mathcal{O}} + \frac{t\bar{\sigma}}{2} A^{LN}\gamma_{\mathcal{O}}^{-1}\right).$$

Since $\mathcal{O} \neq \mathcal{O}^{-1}$, we have $\gamma_{\mathcal{O}} \neq \gamma_{\mathcal{O}}^{-1}$. By Lemma 2.1, A is injective and hence $A^{LN}\gamma_{\mathcal{O}} \neq A^{LN}(\gamma_{\mathcal{O}}^{-1}) = (A^{LN}\gamma_{\mathcal{O}})^{-1}$ for each $N \in \mathbb{N}_0$; that is, no character in $\{A^{LN}\gamma_{\mathcal{O}}\}_{N \in \mathbb{N}_0}$ is equal to its own inverse, and

we may apply [16, (4.2), (4.3)] with coefficients $a_N := t\sigma$ for all $N \in \mathbb{N}_0$ (see also [27, Theorem 3.2]). This gives, for all $N \in \mathbb{N}_0$,

$$(6.3) \quad \widehat{\mu}_{t,\mathcal{O}}(1) = 1, \quad \widehat{\mu}_{t,\mathcal{O}}(A^{LN}\gamma_{\mathcal{O}}) = t\sigma/2, \quad \text{and} \quad \widehat{\mu}_{t,\mathcal{O}}(A^{LN}\gamma_{\mathcal{O}}^{-1}) = t\bar{\sigma}/2.$$

Any character $\gamma \notin \{1\} \cup \bigcup_{N \in \mathbb{N}_0} \{A^{LN}\gamma_{\mathcal{O}}, A^{LN}\gamma_{\mathcal{O}}^{-1}\}$ is either not of the form appearing in [16, (4.2)], in which case $\widehat{\mu}_{t,\mathcal{O}}(\gamma) = 0$ by [16, (4.3)], or is of that form with at least two nonzero exponents. In either case,

$$(6.4) \quad |\widehat{\mu}_{t,\mathcal{O}}(\gamma)| \leq t^2/4.$$

For $M \in \mathbb{N}$, set $\nu_M := \frac{1}{M} \sum_{j=0}^{M-1} \mu_{t,\mathcal{O}} \circ T^{-j}$. It follows from the usual Cesàro averaging argument that there exists a strictly increasing sequence $\{M_k\}_{k \in \mathbb{N}}$ such that ν_{M_k} converges in the weak* topology to a T -invariant measure $\nu \in \mathcal{M}(G, T)$ as $k \rightarrow +\infty$ (cf. [51, Theorem 6.9]).

For $m \in \mathbb{N}_0$ with $L \nmid m$, the character $A^m\gamma_{\mathcal{O}}$ cannot be $A^{LN}\gamma_{\mathcal{O}}$ for some $N \in \mathbb{N}_0$ by Lemma 2.1, and cannot be $A^{LN}\gamma_{\mathcal{O}^{-1}}$ since $\mathcal{O} \neq \mathcal{O}^{-1}$. Hence, by (6.3) and (6.4),

$$(6.5) \quad \widehat{\nu}_M(\gamma_{\mathcal{O}}) = \frac{1}{M} \left(\sum_{\substack{L \nmid m \\ 0 \leq m \leq M-1}} \frac{t\sigma}{2} \right) + \frac{1}{M} \left(\sum_{\substack{L \nmid m \\ 0 \leq m \leq M-1}} \widehat{\mu}_{t,\mathcal{O}}(A^m\gamma_{\mathcal{O}}) \right) = \frac{1}{M} \cdot \left\lceil \frac{M}{L} \right\rceil \cdot \frac{t\sigma}{2} + z_M,$$

where $z_M \in \mathbb{C}$ and $|z_M| \leq \frac{t^2}{4}$ by (6.4). In the same way, $\widehat{\nu}_M(\gamma_{\mathcal{O}^{-1}}) = (1/M) \cdot \lceil M/L \rceil \cdot t\bar{\sigma}/2 + z'_M$, where $z'_M \in \mathbb{C}$ and $|z'_M| \leq \frac{t^2}{4}$. For $\gamma \notin \{1\} \cup \mathcal{O} \cup \mathcal{O}^{-1}$, each $A^m\gamma$ with $m \in \mathbb{N}_0$ does not belong to any of the equivalence classes $\{1\}$, $\mathcal{O}$, or $\mathcal{O}^{-1}$. Hence by (6.4), $|\widehat{\nu}_M(\gamma)| = |\frac{1}{M} \sum_{m=0}^{M-1} \widehat{\mu}_{t,\mathcal{O}}(A^m\gamma)| \leq \frac{t^2}{4}$.

By weak* convergence, $\widehat{\nu}_{M_k}(\gamma) \rightarrow \widehat{\nu}(\gamma)$ for each $\gamma \in \Gamma$ as $k \rightarrow +\infty$. Hence we have $\widehat{\nu}(\gamma_{\mathcal{O}}) = t\sigma/(2L) + z_0$ and $\widehat{\nu}(\gamma_{\mathcal{O}^{-1}}) = t\bar{\sigma}/(2L) + z'_0$ for some $z_0, z'_0 \in \mathbb{C}$ with $|z_0|, |z'_0| \leq \frac{t^2}{4}$. Moreover, $|\widehat{\nu}(\gamma)| \leq \frac{t^2}{4}$ for each $\gamma \notin \{1\} \cup \mathcal{O} \cup \mathcal{O}^{-1}$.

Since f is real-valued, $\widehat{f}(\gamma) = \overline{\widehat{f}(\gamma^{-1})}$ and thus $b_{\mathcal{O}^{-1}}(f) = \overline{b_{\mathcal{O}}(f)}$. Hence,

$$(6.6) \quad \begin{aligned} \int f d\nu - \int f dm_G &= \sum_{\mathcal{O}' \in \mathcal{O} \setminus \{\mathcal{O}_0\}} b_{\mathcal{O}'}(f) \widehat{\nu}(\gamma_{\mathcal{O}'}^{-1}) \\ &= b_{\mathcal{O}}(f) \left(\frac{t\bar{\sigma}}{2L} + z'_0 \right) + b_{\mathcal{O}^{-1}}(f) \left(\frac{t\sigma}{2L} + z_0 \right) + \sum_{\mathcal{O}' \in \mathcal{O} \setminus \{\mathcal{O}, \mathcal{O}^{-1}, \mathcal{O}_0\}} b_{\mathcal{O}'}(f) \widehat{\nu}(\gamma_{\mathcal{O}'}^{-1}) \\ &= \frac{t}{L} |b_{\mathcal{O}}(f)| + b_{\mathcal{O}}(f) z'_0 + b_{\mathcal{O}^{-1}}(f) z_0 + \sum_{\mathcal{O}' \in \mathcal{O} \setminus \{\mathcal{O}, \mathcal{O}^{-1}, \mathcal{O}_0\}} b_{\mathcal{O}'}(f) \widehat{\nu}(\gamma_{\mathcal{O}'}^{-1}) \\ &\geq \frac{t}{L} |b_{\mathcal{O}}(f)| - \frac{t^2}{4} \sum_{\mathcal{O}' \in \mathcal{O} \setminus \{\mathcal{O}_0\}} |b_{\mathcal{O}'}(f)|. \end{aligned}$$

Since $b_{\mathcal{O}}(f) \neq 0$, the right-hand side is positive for all sufficiently small $t > 0$.

Case 2. $\mathcal{O} = \mathcal{O}^{-1}$. We first note that $\gamma_{\mathcal{O}}^2 = 1$. Indeed, there exist $m, n \in \mathbb{N}_0$ such that $A^m\gamma_{\mathcal{O}} = A^n\gamma_{\mathcal{O}}^{-1}$. If $m > n$, the injectivity of A^n gives $A^{m-n}\gamma_{\mathcal{O}} = \gamma_{\mathcal{O}}^{-1}$, and hence $A^{2(m-n)}\gamma_{\mathcal{O}} = \gamma_{\mathcal{O}}$, contradicting Lemma 2.1. The case $n > m$ is analogous. Thus $m = n$, and the injectivity of A^m gives $\gamma_{\mathcal{O}}^2 = 1$.

Since f is real-valued and $\mathcal{O} = \mathcal{O}^{-1}$, we have $b_{\mathcal{O}}(f) \in \mathbb{R}$. By Lemma 6.4, choose $L \in \mathbb{N}$ such that $\{A^{LN}\gamma_{\mathcal{O}}\}_{N \in \mathbb{N}_0}$ is dissociate, and set $\sigma := b_{\mathcal{O}}(f)/|b_{\mathcal{O}}(f)| \in \{-1, 1\}$. For $0 < t \leq 1$, we now take

$$\mu_{t,\mathcal{O}} := \prod_{N=0}^{+\infty} (1 + \Re(t\sigma A^{LN}\gamma_{\mathcal{O}})) = \prod_{N=0}^{+\infty} (1 + t\sigma A^{LN}\gamma_{\mathcal{O}}).$$

Here $\gamma_{\mathcal{O}}^2 = 1$, so also $(A^{LN}\gamma_{\mathcal{O}})^2 = 1$ for each $N \in \mathbb{N}_0$; moreover, $A^{LN}\gamma_{\mathcal{O}} \neq 1$ by Lemma 2.1. Since $0 < t \leq 1$ and $\sigma \in \{-1, 1\}$, we have $t\sigma \in [-1, 1]$. We may therefore apply [27, Theorem 3.2], with the coefficient function β in the notation of that theorem given by $\beta(A^{LN}\gamma_{\mathcal{O}}) = t\sigma$ for all $N \in \mathbb{N}_0$. As in Case 1, this gives $\hat{\mu}_{t,\mathcal{O}}(1) = 1$ and $\hat{\mu}_{t,\mathcal{O}}(A^{LN}\gamma_{\mathcal{O}}) = t\sigma$ for each $N \in \mathbb{N}_0$, while $|\hat{\mu}_{t,\mathcal{O}}(\gamma)| \leq t^2$ for each $\gamma \notin \{1\} \cup \{A^{LN}\gamma_{\mathcal{O}} : N \in \mathbb{N}_0\}$.

Repeating the Cesàro averaging argument following (6.4), with $t\sigma$ in place of $t\sigma/2$ and t^2 in place of $t^2/4$, gives a measure $\nu \in \mathcal{M}(G, T)$ such that $\hat{\nu}(\gamma_{\mathcal{O}}) = (t/L)\sigma + z_0$, with $|z_0| \leq t^2$, and $|\hat{\nu}(\gamma)| \leq t^2$ for $\gamma \notin \{1\} \cup \mathcal{O}$. Hence,

$$\int f d\nu - \int f dm_G \geq \frac{t}{L} |b_{\mathcal{O}}(f)| - t^2 \sum_{\mathcal{O}' \in \mathcal{O} \setminus \{\mathcal{O}_0\}} |b_{\mathcal{O}'}(f)|.$$

Since $b_{\mathcal{O}}(f) \neq 0$, the right-hand side is positive for all sufficiently small $t > 0$. $\square$

We can now prove Theorem E.

Proof of Theorem E. Suppose m_G is F -maximizing. By Proposition 6.5, $b_{\mathcal{O}}(F) = 0$ for all nontrivial orbits $\mathcal{O}$. Hence (6.2) gives $\int F d\nu = \hat{F}(1) = \int F dm_G = Q(T, F)$ for each $\nu \in \mathcal{M}(G, T)$. Therefore $\int (F - Q(T, F)) d\nu = 0$ for all $\nu \in \mathcal{M}(G, T)$, so $F - Q(T, F)$ is a weak coboundary. $\square$

Remark 6.6. When T is expanding, as for the doubling map, the Mañé lemma and the full support of m_G show that the conclusion of Theorem E holds for Walters functions F , in the stronger form that $F - Q(T, F)$ is a coboundary; see [6], where the Walters condition is also defined. On the one hand the Wiener algebra obstruction is not a consequence of the Walters-class obstruction, and on the other hand the conclusion of Theorem E is weaker: Proposition 6.7 below exhibits, for the doubling map, an $F \in A(\mathbb{T})$ that is a weak coboundary but not a coboundary; since $m_{\mathbb{T}}$ is F -maximizing with full support, the Mañé lemma then shows that F is not Walters.

Proposition 6.7 (Sharpness of Theorem E). *Let $T: \mathbb{T} \rightarrow \mathbb{T}$ be given by $T(x) = 2x \pmod{1}$, set $\lambda_0 := -1$ and $\lambda_m := m^{-1/2} - (m+1)^{-1/2}$ for $m \in \mathbb{N}$, and define*

$$(6.7) \quad F(x) := \sum_{n=0}^{+\infty} \frac{1}{2^n} \left(-\cos(2\pi(2n+1)x) + \sum_{m=1}^{+\infty} \lambda_m \cos(2\pi(2n+1)2^m x) \right).$$

Then the following hold:

- (i) $F \in C(\mathbb{T}, \mathbb{R}) \cap A(\mathbb{T})$, $\sum_{k \in \mathbb{Z}} |\hat{F}(k)| = 4$, and $\hat{F}(k) \neq 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.
- (ii) $\int F d\nu = 0$ for every $\nu \in \mathcal{M}(\mathbb{T}, T)$; in particular, $Q(T, F) = 0$, every invariant measure is F -maximizing, and F is a weak coboundary.
- (iii) There is no $u \in L^2(\mathbb{T})$ such that $F = u - u \circ T$; in particular, F is not a coboundary.
- (iv) F is not a Walters function.

Proof. (i) Each $k \in \mathbb{N}$ can be written uniquely in the form $k = (2n+1)2^m$ with $n, m \in \mathbb{N}_0$, so (6.7) gives

$$(6.8) \quad \hat{F}(k) = \hat{F}(-k) = (1/2) \cdot 2^{-n} \lambda_m \quad \text{for } k = (2n+1)2^m, \quad \hat{F}(0) = 0.$$

Note that

$$(6.9) \quad \sum_{m=1}^M \lambda_m = 1 - \frac{1}{\sqrt{M+1}} \quad \text{for all } M \in \mathbb{N}, \quad \text{so } \sum_{m=1}^{+\infty} \lambda_m = 1.$$

Since $\lambda_0 = -1$ and $\lambda_m > 0$ for all $m \in \mathbb{N}$, no $\hat{F}(k)$ with $k \neq 0$ vanishes, and $\sum_{m=0}^{+\infty} |\lambda_m| = |\lambda_0| + \sum_{m=1}^{+\infty} \lambda_m = 2$ by (6.9). Thus $\sum_{k \in \mathbb{Z}} |\hat{F}(k)| = \sum_{n=0}^{+\infty} 2^{-n} \sum_{m=0}^{+\infty} |\lambda_m| = 4$. The series in (6.7) therefore converges absolutely and uniformly, so F is continuous and real-valued. This proves (i).

(ii) The orbits under iterations of A on $\mathbb{Z} \setminus \{0\}$ are the sets $\{2^m q : m \in \mathbb{N}_0\}$ with q odd, so by (6.8) the orbit sum through $q = \pm(2n+1)$ with $n \in \mathbb{N}_0$ is

$$b_{\{2^m q : m \in \mathbb{N}_0\}}(F) = \frac{1}{2} \cdot 2^{-n} \sum_{m=0}^{+\infty} \lambda_m = \frac{1}{2} \cdot 2^{-n} (\lambda_0 + 1) = 0$$

by (6.9). Thus $b_{\mathcal{O}}(F) = 0$ for every nontrivial orbit $\mathcal{O}$, and (6.2) gives $\int F d\nu = \widehat{F}(0) = 0$ for all $\nu \in \mathcal{M}(\mathbb{T}, T)$. This proves (ii), where the final part of (ii) follows from the characterisation of weak coboundaries recalled in Subsection 1.2.

(iii) Suppose that $u \in L^2(\mathbb{T})$ satisfies $F = u - u \circ T$. Comparing Fourier coefficients, and using $\widehat{u \circ T}(k) = \widehat{u}(k/2)$ for k even and $\widehat{u \circ T}(k) = 0$ for k odd, we obtain from (6.8) that $\widehat{u}(1) = \widehat{F}(1) = -\frac{1}{2}$ and $\widehat{u}(2^M) - \widehat{u}(2^{M-1}) = \widehat{F}(2^M) = \frac{1}{2} \lambda_M$ for $M \in \mathbb{N}$. Summing and applying (6.9),

$$\widehat{u}(2^M) = -\frac{1}{2} + \frac{1}{2} \sum_{m=1}^M \lambda_m = -\frac{1}{2\sqrt{M+1}} \quad \text{for all } M \in \mathbb{N}_0,$$

so $\sum_{M=0}^{+\infty} |\widehat{u}(2^M)|^2 = \frac{1}{4} \sum_{M=0}^{+\infty} (M+1)^{-1} = +\infty$. On the other hand, Parseval's identity gives $\sum_{k \in \mathbb{Z}} |\widehat{u}(k)|^2 = \|u\|_{L^2}^2 < +\infty$, a contradiction. This proves (iii).

(iv) Suppose that F were Walters. Since T is expanding, the Mañé lemma (see [6, Theorem 1]) gives $\varphi \in C(\mathbb{T}, \mathbb{R})$ with $F - Q(T, F) \leq \varphi \circ T - \varphi$ on $\mathbb{T}$, with equality on the support of every F -maximizing measure. By (ii), $m_{\mathbb{T}}$ is F -maximizing with $Q(T, F) = 0$. Together with $\text{supp } m_{\mathbb{T}} = \mathbb{T}$, the equality holds everywhere, i.e., $F = \varphi \circ T - \varphi$, making F a coboundary, contrary to (iii). This proves (iv). $\square$

Remark 6.8. In the notation of Proposition 6.7, define, for $x \in \mathbb{T}$,

$$f_1(x) := \sum_{m=0}^{+\infty} \lambda_m e^{2\pi i \cdot 2^m x}.$$

Since $\lambda_m = r_m - r_{m+1}$, where $r_m := m^{-1/2}$ for each $m \in \mathbb{N}$ with the convention $r_0 := 0$, f_1 is exactly the function used in [8, Section 3] to exhibit a weak coboundary that is not a coboundary. Since the orbits of A on $\mathbb{Z} \setminus \{0\}$ are the sets $\{2^m q : m \in \mathbb{N}_0\}$ with q odd, and $f_1(qx) = \sum_{m=0}^{+\infty} \lambda_m e^{2\pi i q 2^m x}$ relocates the frequency support of f_1 to the orbit through q , (6.7) can be written as

$$F(x) = \sum_{n=0}^{+\infty} \frac{1}{2^n} \Re(f_1((2n+1)x)).$$

That is, the function F is obtained by relocating the single-orbit construction of [8, Section 3] to every nontrivial orbit of A and summing. The vanishing $\sum_{m=0}^{+\infty} \lambda_m = 0$ correspondingly becomes the vanishing of every orbit sum $b_{\mathcal{O}}(F)$ rather than a single one, and the non-coboundary argument in the proof of Proposition 6.7 (iii) is the same Parseval obstruction as in [8, Section 3], applied on every orbit.

7. NO CLOSED FORM FOR THE FOURIER COEFFICIENTS

There is no canonical notion of closed form (see [4] for a discussion of this issue). For sequences, we work with the class of P-recursive sequences, defined below (see Definition 7.1). A P-recursive sequence is determined by finitely many initial terms together with a single linear recurrence with polynomial coefficients; this class was studied systematically by Stanley [46]. It is closed under termwise addition and multiplication and under convolution, and it is the setting for the algorithmic proof of identities between combinatorial sums and special functions; see [33, 38]. The class contains those given by several more restrictive notions of closed form: the finite linear combinations of

hypergeometric terms, that is, of sequences whose ratio of successive terms is a rational function of the index, which are the closed forms in the sense of [38, Chapter 8]; the sequences satisfying a linear recurrence with constant coefficients; and the sequences of Taylor coefficients of algebraic functions. A proof that a sequence is not P-recursive therefore excludes a closed form in each of these senses, and it is in this sense that the title of this section should be understood.

Definition 7.1. A sequence $\{c_n\}_{n \in \mathbb{N}_0}$ in $\mathbb{C}$ is called *P-recursive*, or *holonomic* (cf. [53]), if there exist $r \in \mathbb{N}$, polynomials $p_0, p_1, \dots, p_r \in \mathbb{C}[X]$ with $p_r \neq 0$, and $n_0 \in \mathbb{N}_0$ such that for all integers³ $n \geq n_0$,

$$(7.1) \quad \sum_{j=0}^r p_j(n) c_{n+j} = 0.$$

The ordinary generating series $C(z) = \sum_{n=0}^{+\infty} c_n z^n$, in the ring of formal power series over $\mathbb{C}$, is called *D-finite* if there exist $s \in \mathbb{N}$ and polynomials $q_0, q_1, \dots, q_s \in \mathbb{C}[X]$ with $q_s \neq 0$, such that in the sense of formal power series,

$$(7.2) \quad \sum_{j=0}^s q_j(z) C^{(j)}(z) = 0.$$

By [46, Theorem 1.5], the sequence $\{c_n\}_{n \in \mathbb{N}_0}$ is P-recursive if and only if its ordinary generating series $C(z)$ is D-finite.

We write $\mathcal{D}(\mathbb{T}) := C^\infty(\mathbb{T}, \mathbb{C})$ and denote its topological dual by $\mathcal{D}'(\mathbb{T})$. We identify $F \in C(\mathbb{T}, \mathbb{C})$ with the distribution defined by $\langle F, \phi \rangle := \int_{\mathbb{T}} F(x) \phi(x) dm_{\mathbb{T}}(x)$ for $\phi \in C^\infty(\mathbb{T}, \mathbb{C})$, and all differential equations below are understood in $\mathcal{D}'(\mathbb{T})$ (in particular higher derivatives are defined weakly via integration by parts against test functions). Recall that for $F \in C(\mathbb{T}, \mathbb{C})$ and $n \in \mathbb{Z}$, $\widehat{F}(n) = \int_{\mathbb{T}} F(x) e^{-2\pi i n x} dm_{\mathbb{T}}(x)$.

For Fourier coefficients, P-recursive has an analytic formulation analogous to its equivalence with D-finiteness: for $F \in C(\mathbb{T}, \mathbb{R})$, the sequence $\{\widehat{F}(n)\}_{n \in \mathbb{N}_0}$ is P-recursive if and only if F satisfies, in $\mathcal{D}'(\mathbb{T})$, a nontrivial linear differential equation with trigonometric-polynomial coefficients (Lemma 7.2 and Remark 7.3). For example, if $\alpha > 0$ then $u(x) := |\sin \pi x|^\alpha$ satisfies $(1 - \cos(2\pi x)) u' = \alpha \pi \sin(2\pi x) u$ in $\mathcal{D}'(\mathbb{T})$, since u is absolutely continuous and the equation holds on $\mathbb{T} \setminus \{0\}$; so the Fourier coefficients of u are P-recursive. In these terms, Theorems A and F show that the function $\widetilde{F}$ of Theorem A, each of whose Fourier coefficients is rational and is determined exactly by a finite procedure, satisfies no nontrivial linear differential equation with trigonometric-polynomial coefficients.

We wish to prove Proposition 7.11, that a real-valued continuous function with P-recursive Fourier coefficients is Hölder continuous. For this we use the fact that P-recursive of $\{\widehat{F}(n)\}_{n \in \mathbb{N}_0}$ is equivalent to D-finiteness of the associated generating series, and the substitution $z = e^{2\pi i x}$ converts the differential equation in z into a differential equation for F itself, with trigonometric-polynomial coefficients (Lemma 7.2). Such an equation is nonsingular away from the finitely many zeros of its leading coefficient, and there F is real analytic. At a zero the equation is singular, and the local behaviour of its solutions is described by classical asymptotic theory: the formal solutions were determined by Fabry [15], in the generality we need by Cope [14], and they are realised by actual solutions through the asymptotic existence theorem (due in its original form to Birkhoff [2], see Wasow [52] for the generality needed here). On each side of the singular point F is therefore a finite combination of solutions admitting asymptotic expansions built from terms $e^{Q(r)} r^\nu (\log r)^k$, where

³The apparently stronger requirement that (7.1) hold for all $n \in \mathbb{N}_0$, which is the form in which the definition is usually stated, is in fact equivalent. Indeed, if (7.1) holds for all $n \geq n_0$, then replacing each p_j by $\widetilde{p}_j(X) = p_j(X) \prod_{i=0}^{n_0-1} (X - i)$ yields polynomials, still with $\widetilde{p}_r \neq 0$ since $\mathbb{C}[X]$ is an integral domain, for which the recurrence holds for all $n \in \mathbb{N}_0$.

Q is a polynomial in a fractional power of $1/r$. Continuity of F then forces each $\Re Q$ to be bounded above; since $\Re Q$ is a real polynomial in a fractional power of $1/r$ without constant term, only two possibilities remain, that $\Re Q \rightarrow -\infty$, in which case the terms concerned are $\mathcal{O}(r^N)$ for all $N > 0$, or that $\Re Q \equiv 0$. Among the terms of the second kind, continuity forces every surviving exponent ν to have $\Re \nu > 0$. This yields a bound $|F(x_0 \pm r) - F(x_0)| = \mathcal{O}(r^\beta)$ for some $\beta > 0$, together with a polynomial bound on the derivative; interpolating between the two gives Hölder continuity. In both of these steps, the passage from continuity of F to the conclusion uses a non-cancellation estimate for finite sums $\sum_{j=1}^J c_j e^{iH_j(s)} s^{i\lambda_j}$ (Lemma 7.6), proved using a mean-square argument.

Lemma 7.2 (A differential equation for functions with P-recursive Fourier coefficients).

Let $F \in C(\mathbb{T}, \mathbb{R})$. If $\{\widehat{F}(n)\}_{n \in \mathbb{N}_0}$ is P-recursive, then there exist $\ell \in \mathbb{N}$ and real-valued trigonometric polynomials $a_0, a_1, \dots, a_\ell$, with $a_\ell \not\equiv 0$, such that

$$(7.3) \quad \sum_{j=0}^{\ell} a_j(x) \frac{d^j F}{dx^j} = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

Proof. Since $|\widehat{F}(n)| \leq \|F\|_{L^1}$ for each $n \in \mathbb{N}_0$, the sequence $\{\widehat{F}(n)\}_{n \in \mathbb{N}_0}$ is bounded. Hence, the series $F_+ := \sum_{n=0}^{+\infty} \widehat{F}(n) \exp(2\pi i n x)$ converges in $\mathcal{D}'(\mathbb{T})$ and defines a distribution, by the rapid decay of the Fourier coefficients of smooth test functions (cf. [21, Theorem 3.3.9]).

By P-recursiveity of $\{\widehat{F}(n)\}_{n \in \mathbb{N}_0}$, its generating series $C(z) := \sum_{n=0}^{+\infty} \widehat{F}(n) z^n$ is D-finite, so that (7.2) holds for some $q_0, q_1, \dots, q_s$ with $q_s \not\equiv 0$. Write $\theta := z \frac{d}{dz}$. For all $j, n \in \mathbb{N}_0$,

$$\theta(\theta - 1) \cdots (\theta - j + 1) z^n = n(n-1) \cdots (n-j+1) z^n = z^j \frac{d^j}{dz^j} z^n.$$

Hence, by linearity, $z^j C^{(j)}(z) = \theta(\theta - 1) \cdots (\theta - j + 1) C(z)$. Multiplying (7.2) by z^s therefore gives

$$(7.4) \quad \sum_{j=0}^s q_j(z) z^{s-j} \theta(\theta - 1) \cdots (\theta - j + 1) C(z) = 0,$$

an identity of formal power series, in which every operator coefficient is a polynomial in z because $j \leq s$.

We next transfer the formal power series identity to an identity of distributions. Consider the coefficientwise Fourier correspondence $z^n := e^{2\pi i n x}$ for each $n \in \mathbb{N}_0$. For each integer j with $0 \leq j \leq s$, $q_j(z) z^{s-j}$ becomes a trigonometric polynomial and θ becomes $(2\pi i)^{-1} \frac{d}{dx}$. Hence, the left-hand side of (7.4) becomes $\sum_{j=0}^s b_j(x) \frac{d^j F_+}{dx^j}$ for some trigonometric polynomials $b_0, b_1, \dots, b_s$. Only the term $j = s$ of (7.4) contributes to order s , the others having order at most $s-1$, so $b_s(x) = (2\pi i)^{-s} q_s(e^{2\pi i x})$; and $b_s \not\equiv 0$, since a nonzero polynomial has only finitely many roots on the unit circle.

By (7.4) and the definition of F_+ , both F_+ and the coefficients b_j have only nonnegative Fourier frequencies, so for each $n \in \mathbb{N}_0$, the coefficient of $e^{2\pi i n x}$ in $\sum_{j=0}^s b_j \frac{d^j F_+}{dx^j}$ is a finite sum, namely the coefficient of z^n in the left-hand side of (7.4), which vanishes; and for integer $n < 0$ the coefficient vanishes for the same reason of frequency support. Since a distribution on $\mathbb{T}$ is determined by its Fourier coefficients (cf. [44, Chapter VII, Section 1, Theorem I]), a distribution whose Fourier coefficients all vanish is identically zero. We therefore conclude that

$$(7.5) \quad \sum_{j=0}^s b_j(x) \frac{d^j F_+}{dx^j} = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

A distributional differential equation satisfied by F_+ need not be satisfied by F . To obtain (7.3), we use the following fact.

Claim. If $u_1, u_2 \in \mathcal{D}'(\mathbb{T})$ each satisfy a nontrivial linear differential equation with trigonometric-polynomial coefficients, then so does $u_1 + u_2$.

To prove the claim, we recast it in the language of differential operators. Writing $z := e^{2\pi i x}$, the trigonometric polynomials form the integral domain $\mathbb{C}[z, z^{-1}]$, and $\frac{d}{dx}$ maps its field of fractions $K := \mathbb{C}(z)$ into itself. Since $\frac{dz}{dx} = 2\pi i z$, we have $\frac{d}{dx} = 2\pi i z \frac{d}{dz}$ on K . Let R denote the set of operators $\sum_{k=0}^{+\infty} f_k \frac{d^k}{dx^k}$ with $f_k \in K$ and finitely many f_k nonzero. Then R is closed under composition: by the relation $\frac{d}{dx} \circ g = g \frac{d}{dx} + g'$, valid for all $g \in K$, any product of such operators may be rewritten with all coefficients preceding all derivatives, and the coefficients so produced again lie in K . Moreover R has no zero divisors: if $S, S' \in R$ are nonzero, of orders a and b and with leading coefficients f_a and g_b , then the same relation shows that SS' has order $a + b$ and leading coefficient $f_a g_b \neq 0$.

Suppose $u_1, u_2 \in \mathcal{D}'(\mathbb{T})$ each satisfy a nontrivial linear differential equation with trigonometric-polynomial coefficients. Write these equations as $P_1 u_1 = 0$ and $P_2 u_2 = 0$ in $\mathcal{D}'(\mathbb{T})$, where $P_1, P_2 \in R \setminus \{0\}$.

We first show that P_1, P_2 have a nonzero common left multiple in R . For each $n \in \mathbb{N}_0$, let R_n be the left K -vector subspace of R , which consists of those operators of order at most n . Hence, R_n has dimension $n + 1$, with basis $1, \frac{d}{dx}, \dots, \frac{d^n}{dx^n}$. Let d_1, d_2 denote the orders of P_1, P_2 , respectively. Set $d := \max\{d_1, d_2\}$ and take an integer $n \geq d$. For each $i \in \{1, 2\}$, right multiplication by P_i defines a left K -linear map p_i . Since R has no zero divisors, p_i is injective; it therefore embeds R_n into R_{n+d} , giving two subspaces of dimension $n + 1$ inside a space of dimension $n + d + 1$. As $2(n + 1) > n + d + 1$, the two subspaces $R_n P_1$ and $R_n P_2$ meet in some nonzero element, which may be written as $S_1 P_1 = S_2 P_2 \neq 0$ with $S_1, S_2 \in R_n$ nonzero (cf. [53, Proposition 3.1]). Thus, P_1 and P_2 have a nonzero common left multiple in R .

Elements of R have coefficients in the fraction field K , which may have poles on the unit circle, so they need not act on $\mathcal{D}'(\mathbb{T})$; we therefore clear denominators. Choose $g \in \mathbb{C}[z, z^{-1}] \setminus \{0\}$ such that gS_1 and gS_2 both have coefficients in $\mathbb{C}[z, z^{-1}]$ and set $P := (gS_1)P_1 = (gS_2)P_2$. The coefficients of P are trigonometric polynomials, so P does act on $\mathcal{D}'(\mathbb{T})$, and $P \neq 0$ since $g \neq 0$ and $S_1 P_1 \neq 0$. Since $P_1 u_1 = P_2 u_2 = 0$ in $\mathcal{D}'(\mathbb{T})$, we have

$$P(u_1 + u_2) = (gS_1)P_1 u_1 + (gS_2)P_2 u_2 = 0 \quad \text{in } \mathcal{D}'(\mathbb{T}).$$

Since P is nonzero and has trigonometric-polynomial coefficients, this is a nontrivial linear differential equation of the required form, which proves the claim.

Since F is real-valued, $\widehat{F}(-n) = \overline{\widehat{F}(n)}$, and hence $F = F_+ + \overline{F_+} - \widehat{F}(0)$ in $\mathcal{D}'(\mathbb{T})$. Conjugating (7.5) shows that $\overline{F_+}$ satisfies the analogous equation with coefficients $\overline{b_j}$, which are trigonometric polynomials, while the constant $-\widehat{F}(0)$ is annihilated by $\frac{d}{dx}$. Thus, applying the claim twice to the above decomposition of F , we obtain a nontrivial linear differential equation $\sum_{j=0}^L g_j F^{(j)} = 0$, where $L \in \mathbb{N}_0$, $g_0, g_1, \dots, g_L$ are trigonometric polynomials, and $g_L \neq 0$. Set $g_j = \alpha_j + i\beta_j$ for each integer $0 \leq j \leq L$, where α_j, β_j are real-valued trigonometric polynomials. Since F is real-valued, each $F^{(j)}$ is a real-valued distribution. Separating real and imaginary parts gives

$$\sum_{j=0}^L \alpha_j F^{(j)} = 0 \quad \text{and} \quad \sum_{j=0}^L \beta_j F^{(j)} = 0.$$

As $g_L \neq 0$, at least one of α_L, β_L is not identically zero; interchanging the two equations if necessary, we may assume that $\alpha_L \neq 0$.

If $L \geq 1$, then the first equation, with $\ell := L$ and $a_j := \alpha_j$ for each integer $0 \leq j \leq L$, is of the required form. If instead $L = 0$, then $\alpha_0 F = 0$ in $\mathcal{D}'(\mathbb{T})$ with $\alpha_0 \neq 0$; the product $\alpha_0 F$ is continuous, so it vanishes identically, and since a nonzero trigonometric polynomial has only finitely many zeros, F vanishes on a dense subset of $\mathbb{T}$ and hence $F \equiv 0$ by continuity. In that case F satisfies $F' = 0$, which is of the required form with $\ell = 1$. Either way the lemma follows. $\square$

Remark 7.3. The converse of Lemma 7.2 holds, and does not require continuity or real-valuedness. Suppose $u \in \mathcal{D}'(\mathbb{T})$ satisfies $\sum_{j=0}^{\ell} a_j u^{(j)} = 0$ in $\mathcal{D}'(\mathbb{T})$ for trigonometric polynomials $a_0, a_1, \dots, a_{\ell}$, not all identically zero, and write $\widehat{u}(n) := \langle u, e_{-n} \rangle$ for each $n \in \mathbb{Z}$, where $e_{-n}(x) := e^{-2\pi i n x}$. Let k_- and k_+ be the smallest and largest $k \in \mathbb{Z}$ such that $\widehat{a}_j(k) \neq 0$ for some integer j with $0 \leq j \leq \ell$, and set $P_k(X) := \sum_{j=0}^{\ell} \widehat{a}_j(k) (2\pi i (X - k))^j$ for $k \in \mathbb{Z}$. Comparing Fourier coefficients gives

$$\sum_{k=k_-}^{k_+} P_k(n) \widehat{u}(n - k) = 0 \quad \text{for all } n \in \mathbb{Z},$$

and $P_{k_-} \neq 0$, since $\widehat{a}_j(k_-) \neq 0$ for some j and the polynomials $(X - k_-)^j$, $0 \leq j \leq \ell$, are linearly independent. If $k_+ > k_-$, the substitution $n := m + k_+$ turns this equation into a recurrence of the form (7.1) for $\{\widehat{u}(m)\}_{m \in \mathbb{N}_0}$, of order $k_+ - k_-$, with leading coefficient $P_{k_-}(X + k_+) \neq 0$, and valid for all $m \in \mathbb{N}_0$. If $k_+ = k_-$, it gives $\widehat{u}(n) = 0$ for all sufficiently large n . In either case $\{\widehat{u}(n)\}_{n \in \mathbb{N}_0}$ is P-recursive.

The equation obtained in Lemma 7.2 is singular at the zeros of its leading coefficient, of which there are only finitely many, since a nonzero trigonometric polynomial has only finitely many zeros in $\mathbb{T}$. On every open interval containing no such zero, the following elementary lemma applies, showing that any distributional solution there is a classical one, that is, a function differentiable as many times as the order of the equation and satisfying it pointwise. On an interval, the classical solutions of an equation of order $m \in \mathbb{N}$ with nowhere-vanishing leading coefficient form a vector space of dimension m , and a basis of it is called a *fundamental set*.

Lemma 7.4 (Regularity at nonsingular points). *Let $J \subseteq \mathbb{R}$ be an open interval, let $m \in \mathbb{N}$, and let $b_0, b_1, \dots, b_m \in C^\infty(J, \mathbb{C})$ with b_m nowhere zero. If $v \in \mathcal{D}'(J)$ satisfies $\sum_{j=0}^m b_j v^{(j)} = 0$ in $\mathcal{D}'(J)$, then $v \in C^\infty(J, \mathbb{C})$ and v is a classical solution (i.e., satisfies the equation pointwise); in particular v lies in the span of any fundamental set on J .*

Proof. Dividing by b_m we may assume $b_m \equiv 1$. Set $V := (v, v', \dots, v^{(m-1)})^t$, a vector of distributions on J , and write

$$M := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -b_0 & -b_1 & \cdots & -b_{m-1} \end{pmatrix},$$

a matrix with smooth entries, and it is directly verified that $V' = MV$. Let $\Phi \in C^\infty(J, \text{GL}_m(\mathbb{C}))$ be a classical fundamental matrix for the system, that is, a matrix-valued solution of $\Phi' = M\Phi$ whose columns are linearly independent at every point, so that $\det \Phi$ is nowhere zero; such a Φ exists on the whole of J : solutions of a linear system with continuous coefficients extend across the whole interval on which the system is defined, and taking the solutions with initial values $e_1, e_2, \dots, e_m \in \mathbb{C}^m$ at a point of J gives a Φ whose determinant is nowhere zero, by Abel's formula (see e.g. [48]). Since Φ and Φ^{-1} are smooth, the product $\Phi^{-1}V$ is a well-defined vector of distributions, and by the product rule together with $(\Phi^{-1})' = -\Phi^{-1}\Phi'\Phi^{-1}$,

$$(\Phi^{-1}V)' = -\Phi^{-1}\Phi'\Phi^{-1}V + \Phi^{-1}V' = -\Phi^{-1}M\Phi\Phi^{-1}V + \Phi^{-1}MV = 0.$$

A distribution whose derivative vanishes on an interval is constant there, so $\Phi^{-1}V = c$ for some $c \in \mathbb{C}^m$ and $V = \Phi c$. In particular $v = (\Phi c)_1$ is smooth and is a classical solution. $\square$

The next four lemmas are independent of differential equations. The first (Lemma 7.5) is an elementary comparison of exponential factors, the second (Lemma 7.6) gives a non-cancellation estimate, the third (Lemma 7.7) combines them into the form in which they are applied at a

singular point in the proof of Lemma 7.10, and the fourth (Lemma 7.8) is a calculus lemma used at the end of that proof.

We use the convention that the *sign* of a real number t is -1 , 0 , or 1 according as $t < 0$, $t = 0$, or $t > 0$.

Lemma 7.5 (Comparison of exponential factors). *Let $q, J \in \mathbb{N}$ and let $Q_1, Q_2, \dots, Q_J$ be polynomials in $r^{-1/q}$ without constant term. Then there is $r_0 \in (0, 1)$ such that, for each pair of integers $i, i' \in [1, J]$, the sign of $\Re Q_i(r) - \Re Q_{i'}(r)$ is constant on $(0, r_0)$. Moreover, this sign is zero on $(0, r_0)$ if and only if $\Re Q_i \equiv \Re Q_{i'}$. Consequently, writing $\Psi(r) := \max\{\Re Q_i(r) : i \in \mathbb{N}, i \leq J\}$, the set*

$$(7.6) \quad I^* := \{i \in \mathbb{N} : i \leq J, \Re Q_i(r) = \Psi(r) \text{ for all sufficiently small } r > 0\}$$

is nonempty, and the indices split into two classes as follows. For $i \in I^*$ we have $\Re Q_i = \Psi$ throughout $(0, r_0)$. For $i \notin I^*$ we have $\Re Q_i(r) - \Psi(r) \rightarrow -\infty$ as $r \downarrow 0$, and moreover $e^{\Re Q_i(r) - \Psi(r)} = \mathcal{O}(r^\tau)$ as $r \downarrow 0$, for every $\tau > 0$.

Proof. For each pair of integers $i, i' \in [1, J]$, $\Re(Q_i - Q_{i'})$ is a real polynomial in $s := r^{-1/q}$ without constant term. So as $r \downarrow 0$, $\Re(Q_i - Q_{i'})$ tends to $+\infty$, tends to $-\infty$, or vanishes identically, according to the sign of the leading coefficient or its being the zero polynomial; in particular its sign is constant below some threshold $r_{i,i'} > 0$, and is zero throughout exactly when $\Re Q_i \equiv \Re Q_{i'}$. Set $r_0 := \min(\{1/2\} \cup \{r_{i,i'} : i, i' \in \mathbb{N}, 1 \leq i < i' \leq J\}) \in (0, 1)$. On $(0, r_0)$, the relative order of the functions $\Re Q_1, \Re Q_2, \dots, \Re Q_J$ is fixed, so the maximum Ψ is attained throughout $(0, r_0)$ by one and the same nonempty set of indices, which is therefore exactly the set I^* of (7.6). Fixing $i^* \in I^*$, for $i \notin I^*$ the difference $\Re Q_i - \Psi = \Re(Q_i - Q_{i^*})$ is a nonzero real polynomial in s without constant term which is negative on $(0, r_0)$, and hence tends to $-\infty$ as $r \downarrow 0$. More precisely, the exponential of a polynomial in $r^{-1/q}$ with negative leading coefficient decays faster than every positive power of r as $r \downarrow 0$. $\square$

Next we establish a non-cancellation estimate, proved by a mean-square argument.

Lemma 7.6 (Mean-square non-cancellation). *Let $J \in \mathbb{N}$, let $H_1, H_2, \dots, H_J$ be real polynomials with zero constant term, and let $\lambda_1, \lambda_2, \dots, \lambda_J \in \mathbb{R}$, and suppose that the pairs (H_j, λ_j) are pairwise distinct. Then, for all $c_1, c_2, \dots, c_J \in \mathbb{C}$, the function $S(s) := \sum_{j=1}^J c_j e^{iH_j(s)} s^{i\lambda_j}$ satisfies*

$$\lim_{R \rightarrow +\infty} \frac{1}{\log R} \int_1^R |S(s)|^2 \frac{ds}{s} = \sum_{j=1}^J |c_j|^2.$$

In particular, if $S(s) \rightarrow 0$ as $s \rightarrow +\infty$ then $c_1 = c_2 = \dots = c_J = 0$.

Proof. It suffices to show that for a real polynomial H with zero constant term and $\mu \in \mathbb{R}$ with $(H, \mu) \neq (0, 0)$,

$$(7.7) \quad \lim_{R \rightarrow +\infty} \frac{1}{\log R} \int_1^R e^{iH(s)} s^{i\mu} \frac{ds}{s} = 0.$$

The lemma then follows by expanding $|S|^2$. The diagonal terms contribute $\sum_{j=1}^J |c_j|^2$, while the off-diagonal terms are handled by (7.7) with $H := H_j - H_k$ and $\mu := \lambda_j - \lambda_k$ for integers $1 \leq j, k \leq J$ with $j \neq k$. For such j and k , $H_j - H_k$ is a real polynomial with zero constant term, and $(H_j - H_k, \lambda_j - \lambda_k) \neq (0, 0)$ because the pairs (H_j, λ_j) are distinct.

If H is constant, then $H \equiv 0$ and $\mu \neq 0$. Hence $\int_1^R s^{i\mu-1} ds = (R^{i\mu} - 1)/(i\mu)$ is bounded. Otherwise, denote $d := \deg H \geq 1$. Since H' is a nonzero polynomial, there exists $R_0 > 1$ such that $H'(s) \neq 0$ for all $s \geq R_0$. There is a constant $C > 0$ such that, for all $s \geq R_0$,

$$(7.8) \quad \left| \frac{s^{i\mu-1}}{H'(s)} \right| \leq C s^{-d} \quad \text{and} \quad \left| \frac{d}{ds} \left(\frac{s^{i\mu-1}}{H'(s)} \right) \right| \leq C s^{-d-1}.$$

Since $e^{\mathbf{i}H(s)} s^{\mathbf{i}\mu} \frac{ds}{s} = \frac{s^{\mathbf{i}\mu-1}}{\mathbf{i}H'(s)} d(e^{\mathbf{i}H(s)})$, integration by parts gives

$$(7.9) \quad \int_{R_0}^R e^{\mathbf{i}H(s)} s^{\mathbf{i}\mu} \frac{ds}{s} = \frac{e^{\mathbf{i}H(s)} s^{\mathbf{i}\mu-1}}{\mathbf{i}H'(s)} \Big|_{R_0}^R - \int_{R_0}^R e^{\mathbf{i}H(s)} \frac{d}{ds} \left(\frac{s^{\mathbf{i}\mu-1}}{\mathbf{i}H'(s)} \right) ds.$$

By (7.8), the boundary term in (7.9) is bounded; also the remaining integrand is $\mathcal{O}(s^{-d-1})$ with $d+1 \geq 2$, so the integral term is also bounded. Hence the right-hand side of (7.9) is $\mathcal{O}(1)$. Adding the contribution from $[1, R_0]$, a constant independent of R , and dividing by $\log R$, gives (7.7). $\square$

The following consequence of Lemmas 7.5 and 7.6 isolates the dominant term of a finite sum of exponential-power-logarithmic terms, of the kind occurring in the asymptotic expansions below. The difficulty it addresses is that several Q_i may share the largest real part without being equal, since $\Re Q_i \equiv \Re Q_{i'}$ forces only that $Q_i - Q_{i'}$ be purely imaginary. After factoring out a dominant exponential factor as in Lemma 7.5, and then the dominant power and logarithmic factors, the surviving terms have constant nonzero modulus and differ only in phase; Lemma 7.6 prevents these oscillatory terms from cancelling.

In the following, the imaginary part of a complex number z is denoted by $\Im(z)$.

Lemma 7.7 (The leading term of a finite exponential sum). *Let $q, J \in \mathbb{N}$, and let $(Q_1, \nu_1, k_1), (Q_2, \nu_2, k_2), \dots, (Q_J, \nu_J, k_J)$ be pairwise distinct triples, with each Q_i a polynomial in $r^{-1/q}$ without constant term (possibly $Q_i \equiv 0$), $\nu_i \in \mathbb{C}$, and $k_i \in \mathbb{N}_0$, and suppose $c_1, c_2, \dots, c_J \in \mathbb{C} \setminus \{0\}$. Set*

$$(7.10) \quad w(r) := \sum_{i=1}^J c_i e^{Q_i(r)} r^{\nu_i} (\log r)^{k_i} \quad \text{for } 0 < r < 1.$$

Let Ψ and I^* be as in Lemma 7.5, formed from $Q_1, Q_2, \dots, Q_J$, so that $I^* \neq \emptyset$. Define

$$(7.11) \quad \rho^* := \min\{\Re \nu_i : i \in I^*\} \in \mathbb{R} \quad \text{and} \quad K^* := \max\{k_i : i \in I^*, \Re \nu_i = \rho^*\} \in \mathbb{N}_0.$$

Then there exists a constant $c > 0$ and a sequence $\{r_l\}_{l \in \mathbb{N}}$ of points in $(0, 1)$ with $r_l \rightarrow 0$ as $l \rightarrow +\infty$ such that

$$(7.12) \quad |w(r_l)| \geq c e^{\Psi(r_l)} r_l^{\rho^*} |\log r_l|^{K^*} \quad \text{for all } l \in \mathbb{N}.$$

Proof. Let r_0 be as in Lemma 7.5, fix $i^* \in I^*$, and set $Q^* := Q_{i^*}$, so that $\Re Q^* = \Psi$ on $(0, r_0)$, and $\Re Q_i \equiv \Re Q^*$ holds precisely for $i \in I^*$.

Divide $w(r)$ by $e^{Q^*(r)} r^{\rho^*} |\log r|^{K^*}$. A term in $w(r)$ with $\Re Q_i \neq \Re Q^*$ acquires the factor $e^{(Q_i - Q^*)(r)}$, of modulus $e^{\Re(Q_i - Q^*)(r)}$, which by Lemma 7.5 tends to 0 faster than every power of r . Since the remaining factor $r^{\nu_i - \rho^*} (\log r)^{k_i} |\log r|^{-K^*}$ is bounded in modulus by a power of r^{-1} , the whole term tends to 0 as $r \downarrow 0$. By the same elementary comparison, a term in $w(r)$ with $\Re Q_i \equiv \Re Q^*$ but $\Re \nu_i > \rho^*$, or with $\Re Q_i \equiv \Re Q^*$, $\Re \nu_i = \rho^*$ but $k_i < K^*$, acquires a factor tending to 0 by a power of r or of $\log r$, respectively. What remains is

$$(7.13) \quad \frac{w(r)}{e^{Q^*(r)} r^{\rho^*} |\log r|^{K^*}} = \sum_{i \in I_0} c'_i e^{\mathbf{i}\Pi_i(s)} s^{\mathbf{i}\mu_i} + o(1),$$

where $s := r^{-1/q}$ and $I_0 := \{i \in I^* : \Re \nu_i = \rho^*, k_i = K^*\}$; each c'_i is a nonzero constant multiple of c_i , namely $c'_i = (-1)^{K^*} c_i$; $\mu_i := -q\Im \nu_i$; and $\Pi_i := \Im(Q_i - Q^*)$, a real polynomial in s without constant term, the relevance of $\Re(Q_i - Q^*) \equiv 0$ for $i \in I_0$ being that it makes $e^{(Q_i - Q^*)(r)} = e^{\mathbf{i}\Pi_i(s)}$. For all $i \in I_0$, the pairs (Π_i, μ_i) are pairwise distinct. Indeed, suppose that $(\Pi_i, \mu_i) = (\Pi_{i'}, \mu_{i'})$ for some distinct $i, i' \in I_0$. Then $\Pi_i = \Pi_{i'}$ implies $\Im Q_i = \Im Q_{i'}$, which together with $\Re Q_i = \Re Q_{i'}$ gives $Q_i = Q_{i'}$. Equality of μ_i and $\mu_{i'}$ then gives $\Im \nu_i = \Im \nu_{i'}$, which together with $\Re \nu_i = \Re \nu_{i'} = \rho^*$ gives $\nu_i = \nu_{i'}$. Since $k_i = k_{i'} = K^*$, the triples (Q_i, ν_i, k_i) and $(Q_{i'}, \nu_{i'}, k_{i'})$ would coincide, a contradiction.

Hence, Lemma 7.6 applies to $S(s) := \sum_{i \in I_0} c'_i e^{i\text{III}_i(s)} s^{i\mu_i}$, the sum on the right-hand side of (7.13). Writing $\Sigma := \sum_{i \in I_0} |c'_i|^2$, a positive number since I_0 is nonempty by the definitions of ρ^* and K^* and $c'_i \neq 0$ for each $i \in I_0$, we obtain that

$$(7.14) \quad \frac{1}{\log R} \int_1^R |S(s)|^2 \frac{ds}{s} \longrightarrow \Sigma \quad \text{as } R \rightarrow +\infty.$$

Consequently, $|S(s)|$ cannot remain smaller than $\frac{1}{2}\sqrt{\Sigma}$ for all sufficiently large s . Otherwise, there would exist $s_0 > 1$ such that $|S(s)| \leq \frac{1}{2}\sqrt{\Sigma}$ for all $s \geq s_0$. Then for $R > s_0$, the left-hand side of (7.14) would be at most $\frac{1}{\log R} \int_1^{s_0} |S|^2 \frac{ds}{s} + \frac{1}{4}\Sigma$, which tends to $\frac{1}{4}\Sigma < \Sigma$ as $R \rightarrow +\infty$, contradicting (7.14).

Hence, there exists a sequence $\{s_l\}_{l \in \mathbb{N}}$ in $[1, +\infty)$ such that $s_l \rightarrow +\infty$ as $l \rightarrow +\infty$ and $|S(s_l)| \geq \frac{1}{2}\sqrt{\Sigma}$ for all $l \in \mathbb{N}$. Writing $r_l := s_l^{-q}$, we have $r_l \rightarrow 0$ as $l \rightarrow +\infty$. Since the error term in (7.13) is $o(1)$ as $r \downarrow 0$, its absolute value at $r = r_l$ is smaller than $\frac{1}{4}\sqrt{\Sigma}$ once l exceeds some $l_0 \in \mathbb{N}$. Therefore,

$$\frac{|w(r_l)|}{e^{\Re Q^*(r_l)} r_l^{\rho^*} |\log r_l|^{K^*}} \geq \frac{1}{4}\sqrt{\Sigma} \quad \text{for } l > l_0.$$

Discarding the terms r_l with $l \leq l_0$ and those with $r_l \geq r_0$, on which $e^{\Re Q^*}$ and e^Ψ might differ, and relabelling the remainder as $r_1, r_2, \dots$, we see that (7.12) holds with $c := \frac{1}{4}\sqrt{\Sigma}$. $\square$

The following is an elementary interpolation, deducing Hölder continuity from a decay estimate at a point and a growth estimate on the derivative.

Lemma 7.8 (Hölder continuity by interpolation). *Let $x_0 \in \mathbb{R}$ and $\eta_0 \in (0, 1/2]$, and let $g: (x_0 - \eta_0, x_0 + \eta_0) \rightarrow \mathbb{C}$ be continuous, and continuously differentiable on $(x_0 - \eta_0, x_0) \cup (x_0, x_0 + \eta_0)$. For $\sigma \in \{-1, 1\}$, define $g_\sigma: [0, \eta_0) \rightarrow \mathbb{C}$ by $g_\sigma(r) := g(x_0 + \sigma r) - g(x_0)$, so that $g_\sigma(0) = 0$. Suppose there are $C > 0$, $\beta \in (0, 1)$, and $M \in \mathbb{N}$ such that*

$$(7.15) \quad |g_\sigma(r)| \leq Cr^\beta \quad \text{and} \quad |g'_\sigma(r)| \leq Cr^{-M}$$

for each sign $\sigma \in \{-1, 1\}$ and each $r \in (0, \eta_0)$. Then $\alpha := \beta/(\beta + M) \in (0, \beta)$ satisfies

$$|g(x) - g(y)| \leq 3C|x - y|^\alpha \quad \text{for all } x, y \in (x_0 - \eta_0, x_0 + \eta_0).$$

Proof. Since $\beta + M > 1$, we have $\alpha = \beta/(\beta + M) < \beta$ and $\alpha > 0$. Any two points of $(x_0 - \eta_0, x_0 + \eta_0)$ are at distance at most $2\eta_0 \leq 1$; consequently $h^\beta \leq h^\alpha$ for every distance h arising below.

Fix $\sigma \in \{-1, 1\}$, let $0 \leq s < t < \eta_0$, and set $h := t - s$. If $s = 0$, then $|g_\sigma(t) - g_\sigma(0)| = |g_\sigma(t)| \leq Ct^\beta \leq Ch^\alpha$. If $t \geq 2s > 0$, then $s \leq h$ and $t \leq 2h$, so $|g_\sigma(t) - g_\sigma(s)| \leq C(t^\beta + s^\beta) \leq 3Ch^\beta \leq 3Ch^\alpha$.

It remains to consider the case $0 < t < 2s$. Then $|g_\sigma(t) - g_\sigma(s)| \leq \min\{3Cs^\beta, Chs^{-M}\}$, the first bound because $t^\beta \leq (2s)^\beta \leq 2s^\beta$, the second by integrating (7.15) over $[s, t]$. If $h \geq s^{\beta+M}$, then $s^\beta \leq h^{\beta/(\beta+M)} = h^\alpha$, and the first bound gives $|g_\sigma(t) - g_\sigma(s)| \leq 3Ch^\alpha$. If $h < s^{\beta+M}$, then $h^{1-\alpha} < s^{(\beta+M)(1-\alpha)} = s^M$, so $hs^{-M} < h^\alpha$, and the second bound gives $|g_\sigma(t) - g_\sigma(s)| \leq Ch^\alpha$.

Since $g_\sigma(t) - g_\sigma(s) = g(x_0 + \sigma t) - g(x_0 + \sigma s)$, this covers every pair of points lying strictly on one side of x_0 . The case $s = 0$ covers pairs one of which is x_0 . Finally, for all $r, t > 0$ with $x_0 + r$ and $x_0 - t$ in $(x_0 - \eta_0, x_0 + \eta_0)$, the first estimate in (7.15) gives

$$|g(x_0 + r) - g(x_0 - t)| \leq C(r^\beta + t^\beta) \leq 2C(r + t)^\alpha = 2C|(x_0 + r) - (x_0 - t)|^\alpha. \quad \square$$

The following lemma uses the classical local theory of linear differential equations at a singular point (specifically, Fabry's theorem on formal solutions, and the asymptotic existence theorem realising them as genuine solutions).

Lemma 7.9 (Local structure of solutions at a singular point). *Let $I \subseteq \mathbb{R}$ be an open interval, $x_0 \in I$, $m \in \mathbb{N}$, and $a_0, a_1, \dots, a_m$ be real analytic functions on I with a_m vanishing at x_0 and nowhere else in I . Put $L := a_m(x) \frac{d^m}{dx^m} + \dots + a_1(x) \frac{d}{dx} + a_0(x)$, and fix $\sigma \in \{-1, 1\}$. Then there are*

$q \in \mathbb{N}$, $\eta > 0$, formal expressions $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_m$, and classical solutions $z_1, z_2, \dots, z_m$ of $Lz = 0$ on $I_\sigma^\eta := \{x \in \mathbb{R} : 0 < \sigma(x - x_0) < \eta\}$, a subinterval of I , with the following properties, in which $\zeta_j(r) := z_j(x_0 + \sigma r)$.

(i) Each $\tilde{z}_j$ has the form

$$(7.16) \quad \tilde{z}_j := e^{Q_j(r)} r^{\kappa_j} \sum_{k=0}^{N_j} \sum_{i=0}^{+\infty} b_{k,i}^{(j)} r^{i/q} (\log r)^k,$$

with $N_j \in \mathbb{N}_0$, $\kappa_j \in \mathbb{C}$, $b_{k,i}^{(j)} \in \mathbb{C}$, and Q_j a polynomial in $r^{-1/q}$ without constant term.

(ii) The $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_m$ are linearly independent over $\mathbb{C}$; and expressions built on distinct exponential factors are independent, in the sense that if Q runs over finitely many distinct polynomials in $r^{-1/q}$ without constant term, and for each of them S_Q is a formal sum of terms $r^\nu (\log r)^k$ with $\nu \in \mathbb{C}$ and $k \in \mathbb{N}_0$, then $\sum_Q e^{Q(r)} S_Q(r) = 0$ only if every coefficient of every S_Q vanishes. In particular, every $\mathbb{C}$ -linear combination of $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_m$ and the constant function 1 has, for each triple (Q, ν, k) , a uniquely determined coefficient of $e^{Q(r)} r^\nu (\log r)^k$, obtained by summing all contributions corresponding to that same triple. Terms carrying different exponential factors are never combined.

(iii) $z_1, z_2, \dots, z_m$ form a fundamental set for L on I_σ^η .

(iv) Writing $\tilde{z}_{j,[n]}$ for (7.16) with the inner sum restricted to $0 \leq i \leq n$, for each $\tau > 0$ and each $\ell \in \mathbb{N}_0$, there is $n_0 \in \mathbb{N}$ such that, for all integers $n \geq n_0$ and $1 \leq j \leq m$,

$$(7.17) \quad e^{-Q_j(r)} \partial_r^\ell (\zeta_j(r) - \tilde{z}_{j,[n]}(r)) = \mathcal{O}(r^\tau) \quad \text{as } r \downarrow 0.$$

Proof. It is convenient to move the singular point to the origin. The substitution $x = x_0 + \sigma r$ transforms L into

$$L_\sigma := \sum_{k=0}^m \sigma^k a_k(x_0 + \sigma r) \frac{d^k}{dr^k},$$

whose coefficients are real analytic in r near 0, and whose leading coefficient $\sigma^m a_m(x_0 + \sigma r)$ vanishes at $r = 0$ to the same finite order as a_m does at x_0 , and nowhere else nearby; and z solves $Lz = 0$ on I_σ^η if and only if $r \mapsto z(x_0 + \sigma r)$ solves $L_\sigma \zeta = 0$ on $(0, \eta)$. Being real analytic, the coefficients of L_σ extend holomorphically to a complex neighbourhood of $r = 0$, so that L_σ is a linear differential operator with an isolated singular point at the origin, and the classical local theory at such a point applies to it.

Parts (i) and (ii) are Fabry's theorem on formal solutions, in the form given by Cope [14, Theorem I]: at such a point there are as many linearly independent formal solutions as the order of the equation, each of the form (7.16). The notion of linear independence is [14, Definition 6], that no relation $\sum_{j=1}^m \lambda_j \tilde{z}_j = 0$ holds with constants λ_j not all zero; equivalently, the formal Wronskian does not vanish [14, Theorem IV]. Cope's theorem is proved at $x = \infty$, and the reduction of a finite singular point to that case, together with the resulting form (7.16), is recorded in [14, p. 411] and [47, p. 132, (1.1)]; the uniqueness assertion in (ii) is [14, Definition 1], [47, pp. 132–133].

Parts (iii) and (iv) are the main asymptotic existence theorem, applied on a sector of sufficiently small opening about the positive real r -axis. The method originates with Birkhoff [2], who assumed the characteristic roots distinct, in which case neither fractional powers nor logarithms occur; in the generality needed here, where the formal normal form is that of Hukuhara and Turruttin, we follow Wasow [52, Theorem 19.1], and [47, pp. 140–141] for the statement in the one-sided real form used here. The estimate (7.17) for $\ell \geq 1$ requires no separate argument: the solutions are

produced within a class defined by requiring the asymptotic relation for every derivative [47, p. 135, (2.1)–(2.2)]. Only finitely many ζ_j occur, so a single q may be fixed to serve all of them.⁴ $\square$

The main step in the proof of Proposition 7.11 is Lemma 7.10 below. For the distributional differential equation in Lemma 7.10, away from the zeros of the leading coefficient, standard regularity gives analyticity, so the main issue is the behaviour near a singular point. Lemma 7.9 provides the required local asymptotic expansions. After truncation to arbitrary order, these are finite sums of terms $e^{Q(r)}r^\nu(\log r)^k$, where Q is a polynomial in $r^{-1/q}$ without constant term, $\nu \in \mathbb{C}$, and $k \in \mathbb{N}_0$, together with an error of order $\mathcal{O}(e^{\Re Q(r)}r^\tau)$ for any prescribed $\tau > 0$.

We then separate the exponential factors according as $\Re Q(r) \rightarrow +\infty$, $\Re Q(r) \rightarrow -\infty$, or $\Re Q \equiv 0$ as $r \downarrow 0$. Lemma 7.7 ensures that the dominant terms in such a finite sum cannot cancel completely near 0. Thus, the continuity of the solution u forces terms of the first type, whose absolute values blow up near 0, to vanish. Terms of the second type decay faster than any power of r and are therefore negligible for our purposes. For the remaining terms of the third type, Lemma 7.7 and continuity imply that every surviving exponent ν in $e^{Q(r)}r^\nu(\log r)^k$ satisfies $\Re \nu > 0$. This gives a positive power bound for $u(x) - u(x_0)$, while differentiating the asymptotic expansion gives at most polynomial growth for its derivative. Lemma 7.8 then yields the required Hölder continuity.

Lemma 7.10 (Continuous solutions of singular analytic ODEs). *Let $U \subseteq \mathbb{R}$ be an open interval, let $m \in \mathbb{N}$, and let $a_0, a_1, \dots, a_m$ be real-analytic functions on U with $a_m \not\equiv 0$. Define*

$$L := a_m(x) \frac{d^m}{dx^m} + \dots + a_1(x) \frac{d}{dx} + a_0(x).$$

Set $Z(a_m) := \{x \in U : a_m(x) = 0\}$. If $u \in C(U, \mathbb{C})$ and $Lu = 0$ in $\mathcal{D}'(U)$, then u is locally Hölder continuous. Moreover, u is real analytic on $U \setminus Z(a_m)$.

Proof. Since a_m is analytic and not identically zero, its zeros are isolated and have finite order. On every component of $U \setminus Z(a_m)$, Lemma 7.4 shows that u is a C^∞ classical solution; analyticity of the coefficients then implies that u is real analytic there, since the companion system extends holomorphically to a complex neighbourhood of each such point, where e.g. [48, Theorem 4.1] gives an analytic solution that agrees with u on the real axis by uniqueness.

Fix $x_0 \in Z(a_m)$. After shrinking the interval if necessary, assume that x_0 is the only zero of a_m in $I := (x_0 - \varepsilon, x_0 + \varepsilon)$. Since the same argument applies on both sides of x_0 , it suffices to work on one side of x_0 . Fix $\sigma \in \{-1, 1\}$ and write $I_\sigma := \{x \in I : \sigma(x - x_0) > 0\}$.

Apply Lemma 7.9 on I with this fixed σ , and let $q \in \mathbb{N}$, $\eta > 0$, the formal expressions $\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_m$, and the classical solutions $z_1, z_2, \dots, z_m$ be as provided there; properties (i)–(iv) of that lemma will be used below. As in Lemma 7.9, we write $\zeta_j(r) := z_j(x_0 + \sigma r)$, and $Q_j, \kappa_j, N_j, b_{k,i}^{(j)}$ for the objects attached to $\tilde{z}_j$. Set $v(r) := u(x_0 + \sigma r) - u(x_0)$ for $0 < r < \eta$, so that $v(r) \rightarrow 0$ as $r \downarrow 0$ by continuity. By Lemma 7.4 applied on I_σ , the function u is a classical solution of $Lz = 0$ on I_σ , so by property (iii) of Lemma 7.9,

$$(7.18) \quad v(r) = \sum_{j=1}^m c_j \zeta_j(r) - u(x_0)$$

⁴Wasow's theorem is stated at infinity and for first-order systems: the substitution $\xi = 1/r$ and passage to the companion system of L_σ bring our setting into that form, the companion matrix being meromorphic at $r = 0$ because L_σ has real-analytic coefficients and a leading coefficient vanishing to finite order. Its conclusion is a fundamental matrix $\hat{Y}(\xi) \xi^G e^{\Lambda(\xi)}$ in which Λ is diagonal with entries polynomial in a fractional power of ξ , G is a constant matrix, and $\hat{Y}$ admits an asymptotic expansion in fractional powers of ξ^{-1} . Its columns are the solutions of the companion system, whose first entries are solutions of $L_\sigma \zeta = 0$, and each of these has the form (7.16): the powers of $\log r$ are those produced by $\xi^G = e^{G \log \xi}$ when G is not diagonalisable. In this complex-variable setting, the validity of (7.17) for $\ell \geq 1$ amounts to the term-by-term differentiability of an asymptotic expansion on a sector, cf. [52, Theorem 8.8], which gives the same conclusion independently of [47].

for suitable constants $c_1, c_2, \dots, c_m \in \mathbb{C}$.

For any prescribed constant $\tau > 0$, truncating (7.16) at $i = n$ for some sufficiently large $n \in \mathbb{N}$ expresses ζ_j , for each integer $1 \leq j \leq m$, as a finite $\mathbb{C}$ -linear combination of terms $e^{Q_j(r)} r^\nu (\log r)^k$, with $\nu := \kappa_j + (i/q)$ for one of the finitely many retained pairs (k, i) , together with an error which by (7.17) is $\mathcal{O}(e^{\Re Q_j(r)} r^\tau)$. Note the factor $e^{\Re Q_j}$ in this error, which arises because (7.17) estimates $e^{-Q_j}(\zeta_j(r) - \tilde{z}_{j,[n]}(r))$ rather than $\zeta_j(r) - \tilde{z}_{j,[n]}(r)$ itself: the error is comparable to the terms retained, not small relative to them, so a lower bound on the truncation does not by itself transfer to v . The exponents $\Re Q_j$ must therefore be brought under control first; once the possibility $\Re Q_j(r) \rightarrow +\infty$ has been excluded, the factor $e^{\Re Q_j}$ is bounded and disappears from the error term.

Before carrying out this analysis, we first fix some notation for the coefficients of the expansion of v , and record two facts that will be used throughout. By (7.18), v is asymptotic to the formal expression $\tilde{v} := \sum_{j=1}^m c_j \tilde{z}_j - u(x_0)$, the constant $-u(x_0)$ being regarded, if nonzero, as the trivial term $-u(x_0)e^0 r^0 (\log r)^0$, corresponding to $(Q, \nu, k) = (0, 0, 0)$. By property (ii) of Lemma 7.9, terms carrying different exponential factors are never combined, whereas terms carrying the same one must be. Therefore, for Q a polynomial in $r^{-1/q}$ without constant term, $\nu \in \mathbb{C}$, and $k \in \mathbb{N}_0$, let $C_{Q,\nu,k} \in \mathbb{C}$ denote the total coefficient of $e^{Q(r)} r^\nu (\log r)^k$ in the formal asymptotic expression $\tilde{v}$, obtained by summing the contributions to this term from the indices $j \in \{1, 2, \dots, m\}$ for which $Q_j = Q$, together with $-u(x_0)$ when $(Q, \nu, k) = (0, 0, 0)$. For each fixed Q , the exponents ν occurring lie in the finite union of the sets $\kappa_j + \frac{1}{q}\mathbb{N}_0$ over those j with $Q_j = Q$, so the set of real parts $\Re \nu$ occurring is bounded below. Finally, if $Q \neq 0$ and $c_j \neq 0$ for at least one integer $1 \leq j \leq m$ with $Q_j = Q$, then $\sum_{1 \leq j \leq m: Q_j=Q} c_j \tilde{z}_j \neq 0$ by property (ii) of Lemma 7.9, and hence $C_{Q,\nu,k} \neq 0$ for at least one pair (ν, k) .

We now consider the possible behaviours of the exponential factors. For each integer $1 \leq j \leq m$, $\Re Q_j$ is a real polynomial in $r^{-1/q}$ without constant term. So as $r \downarrow 0$, $\Re Q_j$ either vanishes identically, or tends to $+\infty$, or tends to $-\infty$; in the latter two cases, the behaviour is determined by the sign of its leading nonzero coefficient. We consider these three cases separately.

Claim 1. For all integers $1 \leq j \leq m$ with $c_j \neq 0$, the case $\Re Q_j(r) \rightarrow +\infty$ as $r \downarrow 0$ cannot occur.

Suppose that $\Re Q_{j'}(r) \rightarrow +\infty$ as $r \downarrow 0$ for some integer $1 \leq j' \leq m$ with $c_{j'} \neq 0$. Among the polynomials Q_j with $c_j \neq 0$, together with $Q = 0$ if $u(x_0) \neq 0$, Lemma 7.5 provides one, say Q^* , with $\Re Q_j \leq \Re Q^*$ near 0 for all Q_j with $c_j \neq 0$. Then $\Re Q^*(r) \rightarrow +\infty$ as $r \downarrow 0$ by the assumption on $Q_{j'}$, and in particular $Q^* \neq 0$; since the constant $-u(x_0)$ contributes only to $C_{0,0,0}$, it contributes nothing to any $C_{Q,\nu,k}$ with $Q = Q^*$. Write

$$\mathcal{Q}^* := \{Q_j : j \in \mathbb{N}, j \leq m, c_j \neq 0, \text{ and } \Re Q_j \equiv \Re Q^*\}.$$

Since $Q^* \in \mathcal{Q}^*$ and $Q^* \neq 0$, there is a pair (ν, k) with $C_{Q^*,\nu,k} \neq 0$. Consequently

$$\rho^* := \min\{\Re \nu : \nu \in \mathbb{C}, C_{Q,\nu,k} \neq 0 \text{ for some } Q \in \mathcal{Q}^* \text{ and some } k \in \mathbb{N}_0\}$$

is well defined, since the set on the right is nonempty and is contained in a finite union of arithmetic progressions of the form $\Re \kappa_j + \frac{1}{q}\mathbb{N}_0$ with integer $1 \leq j \leq m$.

Now fix $\tau > \max\{\rho^*, 0\}$. For each integer $1 \leq j \leq m$, every exponent occurring in $\tilde{z}_j$ has the form $\nu = \kappa_j + i/q$ with $i \in \mathbb{N}_0$. Taking the truncations sufficiently long, we may assume that, for each $1 \leq j \leq m$, the remainder $\zeta_j - \tilde{z}_{j,[n]}$ is $\mathcal{O}(e^{\Re Q_j(r)} r^\tau)$, by (7.17) with $\ell = 0$, and that all terms of $\tilde{v}$ with $\Re \nu \leq \rho^*$ are retained. Let w be the finite sum obtained from the retained terms by collecting all contributions corresponding to the same triple (Q, ν, k) , and discarding those triples whose resulting coefficient vanishes. Since all contributions to a triple with $\Re \nu \leq \rho^*$ are retained, the coefficient in w of each such triple is exactly the total coefficient $C_{Q,\nu,k}$. For triples with $\Re \nu > \rho^*$ the truncation may omit some contributions, so their coefficients in w need not be the total coefficients; the argument below does not use them. Since $\Re Q_j(r) \leq \Re Q^*(r)$ for each integer

j with $c_j \neq 0$ and all sufficiently small $r > 0$, the errors above combine to give

$$(7.19) \quad |v(r) - w(r)| \leq C_1 e^{\Re Q^*(r)} r^\tau,$$

for some constant $C_1 > 0$ and all sufficiently small $r > 0$. The triples (Q, ν, k) occurring in w are pairwise distinct and carry nonzero coefficients, and no Q occurring in w has $\Re Q$ exceeding $\Re Q^*$ near 0. Moreover at least one does attain it: the minimum defining ρ^* is attained, so there are $Q \in \mathcal{Q}^*$, ν with $\Re \nu = \rho^*$, and k such that $C_{Q, \nu, k} \neq 0$, and by the choice of the truncations the corresponding coefficient in w is the same nonzero coefficient $C_{Q, \nu, k}$. Hence, the function Ψ of Lemma 7.5, formed from the polynomials occurring in w , agrees with $\Re Q^*$ near 0. The terms indexed by I^* are precisely those with $\Re Q \equiv \Re Q^*$, and the quantity ρ^* that (7.11) then produces is the one defined above. Lemma 7.7, applied to w , therefore yields $K^* \in \mathbb{N}_0$, $c > 0$, and a sequence $r_l \rightarrow 0$ as $l \rightarrow +\infty$ such that $|w(r_l)| \geq c e^{\Re Q^*(r_l)} r_l^{\rho^*} |\log r_l|^{K^*}$. Combining this with (7.19) and using the triangle inequality gives

$$(7.20) \quad |v(r_l)| \geq (c |\log r_l|^{K^*} - C_1 r_l^{\tau - \rho^*}) e^{\Re Q^*(r_l)} r_l^{\rho^*} \geq (1/2) c e^{\Re Q^*(r_l)} r_l^{\rho^*}$$

for all sufficiently large $l \in \mathbb{N}$, where the last inequality holds because $\tau > \rho^*$. Since $\Re Q^*(r)$ is a real polynomial in $r^{-1/q}$ tending to $+\infty$ as $r \downarrow 0$, $e^{-\Re Q^*(r)} = \mathcal{O}(r^N)$ for every $N \in \mathbb{N}$. Hence the right-hand side of (7.20) tends to $+\infty$, contradicting the boundedness of v near 0 by continuity. The claim follows.

By Claim 1 and the preceding classification for $\Re Q_j$, each Q_j with $c_j \neq 0$ satisfies either $\Re Q_j(r) \rightarrow -\infty$ as $r \downarrow 0$ or $\Re Q_j \equiv 0$, and hence $e^{\Re Q_j(r)} \leq 1$ for all sufficiently small $r > 0$. Thus, for every constant $\tau > 0$, truncating (7.16) far enough yields a finite sum $\widehat{w}$, obtained from the retained terms by collecting all contributions corresponding to the same triple (Q, ν, k) , and discarding the vanishing ones, with $v(r) - \widehat{w}(r) = \mathcal{O}(r^\tau)$ and no exponential factor remaining in the error.

We first consider the terms for which $\Re Q(r) \rightarrow -\infty$ as $r \downarrow 0$. For such a Q , the polynomial $\Re Q$ in $r^{-1/q}$ has negative leading coefficient, and hence $e^{\Re Q(r)} = \mathcal{O}(r^N)$ for all $N \in \mathbb{N}$; since $r^\nu (\log r)^k$ is bounded by a power of r^{-1} , each term of $\widehat{w}$ carrying such a factor is $\mathcal{O}(r^N)$ for all $N \in \mathbb{N}$. Now let $\tau > 0$, let $\widehat{w}$ be as above, and let w_0 be the sub-sum of $\widehat{w}$ consisting of its terms with $\Re Q \equiv 0$. Since $\widehat{w}$ is a finite sum, $\widehat{w} - w_0 = \mathcal{O}(r^\tau)$, and together with $v - \widehat{w} = \mathcal{O}(r^\tau)$, this gives $v - w_0 = \mathcal{O}(r^\tau)$.

It remains to consider the terms with $\Re Q \equiv 0$. Set

$$\Xi := \{(Q, \nu, k) : \Re Q \equiv 0 \text{ and } C_{Q, \nu, k} \neq 0\}.$$

Suppose first that $\Xi = \emptyset$, and let $\tau > 0$. Take the truncation of (7.16) long enough that $v - \widehat{w} = \mathcal{O}(r^\tau)$ and all terms with $\Re \nu \leq \tau$ are retained. Then the coefficient in w_0 of every triple (Q, ν, k) with $\Re Q \equiv 0$ and $\Re \nu \leq \tau$ is the total coefficient $C_{Q, \nu, k}$, which vanishes since $\Xi = \emptyset$. Thus, w_0 is a finite linear combination of terms $e^Q r^\nu (\log r)^k$, with $\Re Q \equiv 0$ and $\Re \nu > \tau$, so $w_0 = \mathcal{O}(r^\tau)$, and together with $v(r) - w_0(r) = \mathcal{O}(r^\tau)$ this gives $v = \mathcal{O}(r^\tau)$. Since $\tau > 0$ was arbitrary, (7.21) below holds with every $\gamma > 0$ and $J_0 = 0$.

Suppose therefore that $\Xi \neq \emptyset$, and set $\rho^\dagger := \min\{\Re \nu : (Q, \nu, k) \in \Xi\}$, again a minimum over a set of reals that is bounded below and is contained in a finite union of arithmetic progressions of the form $\Re \kappa_j + \frac{1}{q}\mathbb{N}_0$.

Claim 2. $\rho^\dagger > 0$.

Fix $\tau > \max\{\rho^\dagger, 0\}$, truncate long enough so that every term with $\Re \nu \leq \rho^\dagger$ is retained, and let w_0 be the sub-sum of $\widehat{w}$ consisting of its terms with $\Re Q \equiv 0$. For this truncation, we still have $v - w_0 = \mathcal{O}(r^\tau)$ by the argument above. The sum w_0 is not empty: the minimum defining $\rho^\dagger$ is attained, and for the corresponding triple (Q, ν, k) its coefficient in w_0 is the total coefficient $C_{Q, \nu, k}$, which is nonzero by the definition of Ξ . Since all terms occurring in w_0 have $\Re Q \equiv 0$, the function Ψ of Lemma 7.5, formed from the polynomials occurring in w_0 , vanishes identically, the associated set I^* consists of every index occurring in w_0 , and (7.11) returns the number $\rho^\dagger$ just defined; since

$e^\Psi \equiv 1$, the exponential factor drops out of (7.12). Lemma 7.7 then gives $K^\dagger \in \mathbb{N}_0$, $c' > 0$, and $r'_l \rightarrow 0$ as $l \rightarrow +\infty$ with $|w_0(r'_l)| \geq c' (r'_l)^{\rho^\dagger} |\log r'_l|^{K^\dagger}$. If $\rho^\dagger \leq 0$, the right-hand side does not tend to 0 as $r'_l \downarrow 0$, so neither does $|w_0(r'_l)|$; but $v(r) \rightarrow 0$ as $r \downarrow 0$ and $v - w_0 = \mathcal{O}(r^\tau)$ with $\tau > 0$, so $w_0(r) \rightarrow 0$ as $r \downarrow 0$, a contradiction. The claim follows.

Keep the same τ and w_0 as in the proof of Claim 2. By the choice of the truncation, the coefficient in w_0 of each triple (Q, ν, k) with $\Re Q \equiv 0$ and $\Re \nu < \rho^\dagger$ is the total coefficient $C_{Q, \nu, k}$, which vanishes by the definition of $\rho^\dagger$. Hence w_0 is a finite linear combination of terms $e^{Q(r)} r^\nu (\log r)^k$ with $|e^{Q(r)}| = 1$, $\Re \nu \geq \rho^\dagger > 0$, and $k \in \mathbb{N}_0$, and $v - w_0 = \mathcal{O}(r^\tau)$ with $\tau > 0$. Setting $\gamma := \min\{\rho^\dagger, \tau\} > 0$, there are constants $J_0 \in \mathbb{N}_0$ and $C > 0$ with

$$(7.21) \quad |v(r)| \leq Cr^\gamma (1 + |\log r|)^{J_0}$$

for all sufficiently small $r > 0$. Choose $0 < \beta < \min\{1, \gamma\}$. The logarithmic factors can be absorbed into $r^{\gamma-\beta}$, so there is $C' > 0$ such that $|v(r)| \leq C' r^\beta$ for all sufficiently small $r > 0$.

The derivative of v is handled similarly. By the preceding discussion, each Q_j with $c_j \neq 0$ satisfies either $\Re Q_j \equiv 0$ or $\Re Q_j(r) \rightarrow -\infty$ as $r \downarrow 0$. Hence $|e^{Q_j(r)}| \leq 1$ for all sufficiently small $r > 0$. Fix $\tau > 0$. By (7.18) and Lemma 7.9 (iv) with $\ell = 1$, v' differs from the derivative of a sufficiently long truncation of $\tilde{v}$ by $\mathcal{O}(r^\tau)$. Since

$$\frac{d}{dr} (e^{Q_j(r)} r^\nu (\log r)^k) = e^{Q_j(r)} (Q'_j(r) r^\nu (\log r)^k + \nu r^{\nu-1} (\log r)^k + k r^{\nu-1} (\log r)^{k-1}),$$

the derivative of the truncation is a finite sum of terms $e^{Q_j(r)}$ times a function bounded by a power of r^{-1} . Since $|e^{Q_j(r)}| \leq 1$ for all sufficiently small r , it follows that $v'(r)$ has at most polynomial growth in r^{-1} . Repeating the argument for the other sign gives analogous estimates. After decreasing β , increasing M , shrinking the neighbourhood, and enlarging C if necessary, we may choose common constants $\beta > 0$, $C > 0$, and $M \in \mathbb{N}$ such that, writing $v_\sigma(r) := u(x_0 + \sigma r) - u(x_0)$,

$$(7.22) \quad |v_\sigma(r)| \leq Cr^\beta \quad \text{and} \quad |v'_\sigma(r)| \leq Cr^{-M}$$

for both signs σ and all sufficiently small $r > 0$.

Shrinking η if necessary so that (7.22) holds throughout $(0, \eta)$ and $\eta \leq \frac{1}{2}$, Lemma 7.8 applied to $g := u$ shows that u is Hölder continuous on $(x_0 - \eta, x_0 + \eta)$, with exponent $\alpha := \beta/(\beta + M)$ and constant $3C$. Since the zeros of a_m are isolated and u is analytic away from them, u is locally Hölder continuous on U . $\square$

For $\alpha \in (0, 1]$ we write $C^{0, \alpha}(\mathbb{T}, \mathbb{R})$ for the space of real-valued α -Hölder functions on $\mathbb{T}$, with respect to $d_{\mathbb{T}}$, the quotient metric on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ induced by the Euclidean metric on $\mathbb{R}$.

Proposition 7.11 (Regularity of functions with P-recursive Fourier coefficients). *Let $F \in C(\mathbb{T}, \mathbb{R})$. If $\{\hat{F}(n)\}_{n \in \mathbb{N}_0}$ is P-recursive, then $F \in C^{0, \alpha}(\mathbb{T}, \mathbb{R})$ for some $\alpha \in (0, 1]$, and F is real analytic outside a finite subset of $\mathbb{T}$.*

Proof. By Lemma 7.2, F satisfies (7.3) for some real-valued trigonometric polynomials $a_0, a_1, \dots, a_\ell$ with $a_\ell \neq 0$. In particular, the coefficients are real analytic. Consider F and all the coefficients a_j as 1-periodic functions on $\mathbb{R}$. For $\phi \in C_c^\infty(\mathbb{R}, \mathbb{C})$, the periodisation $\Phi := \sum_{k \in \mathbb{Z}} \phi(\cdot + k)$ lies in $C^\infty(\mathbb{T}, \mathbb{C})$, and, for each integer $j \in [0, \ell]$, $\sum_{k \in \mathbb{Z}} (a_j \phi)^{(j)}(\cdot + k) = (a_j \Phi)^{(j)}$; hence

$$\sum_{j=0}^{\ell} (-1)^j \int_{\mathbb{R}} F(x) (a_j \phi)^{(j)}(x) dx = \sum_{j=0}^{\ell} (-1)^j \int_{\mathbb{T}} F(x) (a_j \Phi)^{(j)}(x) dm_{\mathbb{T}}(x) = 0,$$

where the last equality is (7.3) applied to the test function Φ . Thus (7.3) also holds in $\mathcal{D}'(\mathbb{R})$. Lemma 7.10, applied with $U := \mathbb{R}$ and $m := \ell$, shows that F is locally Hölder continuous on $\mathbb{R}$, hence on $\mathbb{T}$, and that F is real analytic outside the zero set of a_ℓ in $\mathbb{T}$, which is finite since a_ℓ is a

nonzero trigonometric polynomial. Since $\mathbb{T}$ is compact, choose a finite open cover $U_1, U_2, \dots, U_N$ of $\mathbb{T}$ such that, for each $i \in \{1, 2, \dots, N\}$, there exist constants $C_i > 0$ and $\alpha_i \in (0, 1]$ such that

$$|F(x) - F(y)| \leq C_i d_{\mathbb{T}}(x, y)^{\alpha_i}, \quad \text{for all } x, y \in U_i.$$

Set $\alpha := \min\{\alpha_i : i \in \mathbb{N}, i \leq N\}$ and $C_0 := \max\{C_i : i \in \mathbb{N}, i \leq N\}$, and let $\delta \in (0, 1)$ be a Lebesgue number for this cover. Fix $x, y \in \mathbb{T}$. If $d_{\mathbb{T}}(x, y) < \delta$, then x, y belong to some common U_i . Since $d_{\mathbb{T}}(x, y) < 1$ and $\alpha \leq \alpha_i$, we have $|F(x) - F(y)| \leq C_0 d_{\mathbb{T}}(x, y)^{\alpha}$. If $d_{\mathbb{T}}(x, y) \geq \delta$, then $|F(x) - F(y)| \leq 2\|F\|_{\infty} \leq 2\|F\|_{\infty} \delta^{-\alpha} d_{\mathbb{T}}(x, y)^{\alpha}$. Therefore $F \in C^{0, \alpha}(\mathbb{T}, \mathbb{R})$, with Hölder constant at most $\max\{C_0, 2\|F\|_{\infty} \delta^{-\alpha}\}$. $\square$

Remark 7.12. The exponent in Proposition 7.11 cannot be bounded below independently of F , and F need not be real analytic on the whole of $\mathbb{T}$. For $0 < \alpha \leq 1$, the function $u(x) := |\sin \pi x|^{\alpha}$ satisfies $(1 - \cos(2\pi x)) u' = \alpha \pi \sin(2\pi x) u$ in $\mathcal{D}'(\mathbb{T})$, so its Fourier coefficients are P-recursive by Remark 7.3. It is Hölder continuous of exponent α , but not of any larger exponent, and it is real analytic on $\mathbb{T} \setminus \{0\}$ but not at 0.

We can now prove Theorem F.

Proof of Theorem F. The first statement is Proposition 7.11. For the second, let $T \in C^1(\mathbb{T}, \mathbb{T})$ satisfy $|T'| > 1$. By compactness there is $\lambda > 1$ with $|T'| \geq \lambda$ on $\mathbb{T}$, so T is a covering map of $\mathbb{T}$, each of whose inverse branches contracts distances by a factor of at most λ^{-1} . In particular, for all $x, y \in \mathbb{T}$ there is $x' \in T^{-1}(y)$ with $d_{\mathbb{T}}(x, x') \leq d_{\mathbb{T}}(T(x), y)$, which is the weak expansion property of [6, Definition 3]. Moreover, as an expanding circle map, T is topologically transitive. Since F is Hölder continuous and the inverse branches of T are contractions, F satisfies the Walters condition with respect to T , and the Mañé lemma [6, Theorem 1] (see also e.g. [12]) gives $\varphi \in C(\mathbb{T}, \mathbb{R})$ with

$$F - Q(T, F) \leq \varphi \circ T - \varphi \quad \text{on } \mathbb{T}.$$

If μ is an F -maximizing measure, then the continuous function $\varphi \circ T - \varphi - F + Q(T, F)$ is nonnegative and has zero integral with respect to μ , so it vanishes on $\text{supp } \mu$. Taking μ of full support, it vanishes identically, so $F - Q(T, F) = \varphi \circ T - \varphi$ is a coboundary. $\square$

Remark 7.13. In the proof of the second statement of Theorem F, the hypothesis on T is used only to ensure that T is transitive and has the weak expansion property of [6, Definition 3], and that Hölder continuous functions satisfy the Walters condition with respect to T . All three properties hold for all continuous maps $T: \mathbb{T} \rightarrow \mathbb{T}$ that are distance-expanding, in the sense that there exist $\varepsilon > 0$ and $\lambda > 1$ such that $d_{\mathbb{T}}(T(x), T(y)) \geq \lambda d_{\mathbb{T}}(x, y)$ whenever $d_{\mathbb{T}}(x, y) \leq \varepsilon$, so the second statement of Theorem F in fact holds for all such maps.

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